STAPRO: 8272 Model 3G pp. 1–6 (col. fig: NIL)

ARTICLE IN PRESS

Statistics and Probability Letters xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

A proof of the transfer-current theorem in absence of reversibility

L. Avena [a,*], A. Gaudillière [b]

[a] MI, University of Leiden, The Netherlands
[b] Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France

ARTICLE INFO

Article history:
Received 27 March 2018
Received in revised form 19 June 2018
Accepted 20 June 2018
Available online xxxx

MSC:
primary 05C81
15A15
secondary 05C85

Keywords:
Finite networks
Uniform spanning tree
Spanning forests
Determinantal processes
Transfer current theorem

ABSTRACT

The transfer-current theorem is a well-known result in probability theory stating that edges in a uniform spanning tree of an undirected graph form a determinantal process with kernel interpretable in terms of flows. Its original derivation due to Burton and Pemantle (1993) is based on a clever induction using comparison of random walks with electrical networks. Several variants of this celebrated result have recently appeared in the literature. In this paper we give an elementary proof of an extension of this theorem when the underlying graph is directed, irreducible and finite. Further, we give a characterization of the corresponding determinantal kernel in terms of flows extending the kernel given by Burton–Pemantle to the non-reversible setting.

1. Introduction

Given a recurrent connected undirected finite graph $\mathcal{G}$, the Uniform Spanning Tree (UST) is a random spanning tree on the graph sampled uniformly among all possible spanning trees. The analysis of this object can be traced back at least to the work of Kirchhoff (1847) where the number of spanning trees of a graph was characterized in terms of minors of the corresponding discrete Laplacian matrix (matrix-tree theorem). In the last decades, UST has been playing a central role in probability and statistical mechanics due to its deep connection with Markov chain theory and its relation to a number of challenging random combinatorial objects of current interest (e.g. loop erased random walks, percolation, dimers, sandpile models). We refer to Benjamini et al. (2001), Grimmett (2010), Le Jan (2011), Lyons and Peres (2017), Levin et al. (2008) for an overview on the vast literature on the subject. One of the main features making this object particularly interesting is its determinantal nature, namely, local statistics have a closed-form expression in terms of determinants of certain kernels. Determinantal processes are examples of integrable systems (systems allowing for explicit computations) which are currently receiving much attention within the statistical physics and mathematics community, cfr. e.g. Hough et al. (2006) and references therein. In particular, for the UST, the celebrated transfer-current theorem of Burton and Pemantle (1993) states that the corresponding random edges form a determinantal process. That is, for any finite collection of disjoint (undirected) edges $A_k = \{e_1, \dots, e_k\}$ in the graph, we have

$$\mathbb{P}\,(e_1, e_2, \dots, e_k \in UST) = \det\,[\mathcal{J}]_{A_k}, \tag{1.1}$$

* Corresponding author.
E-mail addresses: l.avena@math.leidenuniv.nl (L. Avena), alexandre.gaudilliere@math.cnrs.fr (A. Gaudillière).

https://doi.org/10.1016/j.spl.2018.06.007
0167-7152/

Please cite this article in press as: Avena L., Gaudillière A., A proof of the transfer-current theorem in absence of reversibility. Statistics and Probability Letters (2018), https://doi.org/10.1016/j.spl.2018.06.007.

where $\mathcal{J}$ is the symmetric square matrix with rows and columns indexed by the edges in $\mathcal{G}$, whose entries $\mathcal{J}(e, e')$, for any pair of edges e, e' are defined as follows. Fix an arbitrary orientation of the edges of the given graph $\mathcal{G}$, for edges $e = \overrightarrow{xy}$ and e' of $\mathcal{G}$, $\mathcal{J}(e, e')$ denotes the expected signed number of crossings of e' by the random walk associated to the graph (see (2.1)) started at x and stopped when it hits y. The notation $[\mathcal{J}]_{A_k}$ stands for the matrix $\mathcal{J}$ restricted to its elements doubly indexed in A_k. The matrix $\mathcal{J}$ is often referred to as the *transfer-current kernel*. Several variants and extensions to directed and/or infinite settings of this result have been recently obtained either by using the well-known comparison of Markov chains with electrical networks (see Doyle and Snell, 1984) or by algebraic approaches based e.g. on the Cauchy–Binet formula (see e.g. Pitman and Tang, 2018; Forman, 1993; Kenyon, 2017; Kassel and Wu, 2015). In this paper we give a new proof of the extension of the Burton–Pemantle transfer-current theorem in a directed weighted finite graph. In particular, we derive an original expression for the related kernel generalizing $\mathcal{J}$ in (1.1) and its flow representation.

2. Result: non-reversible transfer-current

Graph. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ be a finite irreducible[1] directed weighted graph, where $\mathcal{V}$ denotes the vertex set, $\mathcal{E}$ the set of oriented edges, and $\mathcal{W} = \{w(e) \geq 0 : e \in \mathcal{E}\}$ a collection of non-negative weights on the (oriented) edges. We denote by $|\mathcal{V}| < \infty$ the corresponding number of vertices. For an oriented edge $e = \overrightarrow{xy} \in \mathcal{E}$, we write $e_- = x$ and $e^+ = y$ for its starting and ending points, respectively.

Random Walk. Associated to the given graph $\mathcal{G}$, we consider the continuous-time random walk $X = (X(t))_{t\geq 0}$ with state space $\mathcal{V}$ and infinitesimal generator

$$(Lf)(x) = \sum_{y\in\mathcal{V}} w(x, y)[f(y) - f(x)], \quad x \in \mathcal{V}, \tag{2.1}$$

for arbitrary $f : \mathcal{V} \to \mathbb{R}$. Let μ be the unique probability measure on $\mathcal{V}$ stationary for the random walk X. Note that the existence and uniqueness is guaranteed by the finiteness and irreducibility assumptions on the graph. We will denote by P_x and E_x, respectively, the law and expectation w.r.t. the random walk X starting at $x \in \mathcal{V}$.

Random Spanning Trees. For $x \in \mathcal{V}$, let $\mathcal{RT}(x) \subset \mathcal{E}$ be the set of spanning trees on $\mathcal{G}$ rooted at x. That is, the set of acyclic connected oriented subgraphs of $\mathcal{G}$ covering $\mathcal{V}$, with a marked point x declared to be the *root* of the tree, where edges are oriented towards this root. We then denote by $\mathcal{RT} = \bigcup_{x\in\mathcal{V}} \mathcal{RT}(x)$ the set of rooted spanning trees on $\mathcal{G}$, and for an element $\tau \in \mathcal{RT}$, by $r[\tau] \in \mathcal{V}$ its root. We can finally define the probability measure on $\mathcal{RT}$ we are interested in, which represents the natural extension of the usual *uniform spanning tree measure* to a directed weighted graph.

Definition 2.1 (*Generalized random spanning tree measure*).
Let $Q := \{q(x) > 0 : x \in \mathcal{V}\}$ be an arbitrary given collection of positive numbers, we call *random spanning tree measure* the probability measure on $\mathcal{RT}$:

$$\nu_Q(\tau) = \frac{w(\tau)q(r[\tau])}{Z_Q}, \quad \tau \in \mathcal{RT} \tag{2.2}$$

where $w(\tau) = \prod_{e\in\tau} w(e)$ is the weight of the tree τ, and $Z_Q = \sum_{\tau\in\mathcal{RT}} w(\tau)q(r[\tau])$ is a normalizing constant. We will denote by τ_Q a random variable with values in $\mathcal{RT}$ and law ν_Q.

Remark 2.2 (*Sampling and Wilson's algorithm*).
Wilson's algorithm, cfr. Wilson (1996), is a by now classical randomized procedure based on killed loop-erased random walks to generate random spanning trees rooted at a given vertex. It follows from the Markov chain tree theorem that τ_Q is the random spanning rooted tree obtained by running Wilson's algorithm (see e.g. Wilson, 1996; Chang, 2013) when the root is chosen with probability $\frac{q(x)\mu(x)}{\sum_z q(z)\mu(z)}$, $x \in \mathcal{V}$. Notice that in case $q(x) \equiv 1$ the root is sampled from the stationary measure μ.

Chang (2013), Thm 5.3.3, has recently showed that τ_Q is a determinantal process but his representation does not allow for a comparison with the original kernel $\mathcal{J}$ in (1.1). The goal of this paper is to fill this gap, we will re-derive the determinantality of τ_Q providing a new proof, but more importantly, we give a kernel interpretable in terms of flows. In order to state our result, let us introduce some notation.

Flows. For x, y in $\mathcal{V}$ and e in $\mathcal{E}$ we define:

$$J_y^+(x, e) = E_x\left[\int_0^{H_y} \mathbb{1}_{\{X(t)=e_-\}} dt\right] w(e), \tag{2.3}$$

as the expected number of crossings of the (oriented) edge e by the process X started from x and stopped at the hitting time of y: $H_y = \inf\{t \geq 0 : X(t) = y\}$. We also define the net flow through e by

$$J_y(x, e) = J_y^+(x, e) - J_y^+(x, -e), \tag{2.4}$$

where $-e = \overrightarrow{e_+e_-}$ for $e = \overrightarrow{e_-e_+}$.

[1] For any pair (x, y) of vertices in $\mathcal{V}$ there exists a direct path of successive neighboring edges starting from x and ending in y.