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The necklace process: A generating function approach

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ABSTRACT

The "necklace process", a procedure constructing necklaces of black and white beads by randomly choosing positions to insert new beads (whose color is uniquely determined based on the chosen location), is revisited. This article illustrates how, after deriving the corresponding bivariate probability generating function, the characterization of the asymptotic limiting distribution of the number of beads of a given color follows as a straightforward consequence within the analytic combinatorics framework.

1. Introduction

We consider the following process (illustrated in Fig. 1) for constructing necklaces with two-colored beads:

- We start with  , the necklace with one black and one white bead.
- New beads can be added between any two adjacent beads. The color of the new bead is determined by the color of those two beads: The new bead is white if and only if its two neighbors are black.

Motivated by a simple network communication model, this "necklace process" was analyzed in Mallows and Shepp (2008). Further variants of this process were discussed in Nakata (2009), where an elegant approach using Pólya urns to model the edges connecting the beads was pursued. The parameters investigated in these articles are the number of white beads in a random necklace of size n (i.e., consisting of n beads), as well as the number of runs of black beads of given length.

The purpose of this brief note is to provide an alternative access to the analysis of the number of beads of a given color in the necklace process by focusing on an appropriate generating function and using tools from analytic combinatorics. A similar approach has been successfully employed in, for example, Morcrette (2012) and Morcrette and Mahmoud (2012).

In Section 2 we briefly discuss the combinatorial structure surrounding the necklace process and elaborate how many different necklaces of given size (i.e., consisting of a given amount of beads) can be constructed. Then, in Section 3 we analyze the number of white and black beads in a random necklace of given size. Our main result is given in Theorem 1, which is an explicit formula for the bivariate probability generating function with respect to the number of white beads—which has a surprisingly nice closed form. Apart from some additional remarks on the structure of this generating function, we then show

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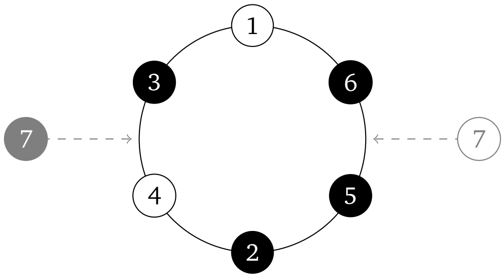

Fig. 1. Construction of a necklace by the necklace process. The numbers on the beads correspond to the order in which they were inserted into the necklace.

in Corollaries 3.1 and 3.2 how the qualitative results concerning the number of black and white beads obtained in Mallows and Shepp (2008) (expectation, variance, limiting distribution) are a straightforward consequence of the explicitly known bivariate generating function.

2. Number of necklaces

Before diving straight into the analysis of the number of beads of a given color, for the sake of completeness, we briefly discuss the combinatorial structure of the objects we are constructing.

While it is rather easy to see that there are $(n-1)!$ possible necklace constructions[1] for a necklace of size n, many of those constructions yield the same necklace. Note that the number of different necklaces of size n is enumerated by sequence A000358 in OEIS (2015). We use the analytic combinatorics framework in order to analyze this quantity in detail.

Proposition 2.1. *Let $\mathcal{N}$ be the combinatorial class containing all different necklaces constructed by the necklace process. The corresponding ordinary generating function $N(z)$ enumerating these necklaces with respect to size is given by*

$$N(z) = \sum_{k\geq 1} \frac{\varphi(k)}{k} \log\left(\frac{1-z^k}{1-z^k-z^{2k}}\right), \tag{1}$$

where $\varphi(k)$ is Euler's totient function. Asymptotically, the number of necklaces of size n is given by

$$[z^n]N(z) = \left(\frac{1+\sqrt{5}}{2}\right)^n n^{-1} + O\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n/2} n^{-1}\right). \tag{2}$$

Proof. The generating function (1) can directly be obtained by means of the machinery provided by the symbolic method (see Chapter I and in particular Theorem I.1 in Flajolet and Sedgewick, 2009). In fact, we can construct the combinatorial class $\mathcal{N}$ as

$$\mathcal{N} = \mathrm{Cyc}(\bigcirc \times \bullet^{+}),$$

where $\bigcirc$ and $\bullet^{+}$ represent the combinatorial classes for a single white and a non-empty sequence of black beads, respectively. Translating the construction of the combinatorial class $\mathcal{N}$ into the language of generating functions then immediately yields (1).

In order to obtain the asymptotic growth of the coefficients of $N(z)$, we use singularity analysis (see Flajolet and Odlyzko, 1990, Flajolet and Sedgewick, 2009, Chapter VI), which requires us to identify the dominant singularities of $N(z)$, i.e., the singularities with minimal modulus.

In fact, by observing that all $\zeta \in \mathbb{C}$ satisfying $\zeta^k = \frac{-1\pm\sqrt{5}}{2}$ are roots of $1 - z^k - z^{2k} = 0$, it is easy to see that $N(z)$ has a unique dominant singularity located at $z = \frac{-1+\sqrt{5}}{2}$ which comes from the first summand of $N(z)$ in (1). Extracting the coefficient growth provided by the first summand and observing that the singularity with the next-larger modulus is located at $z = \sqrt{(-1+\sqrt{5})/2}$ (and comes from the second summand), we obtain (2). $\square$

[1] Starting with $\genfrac{}{}{0pt}{}{\circ}{\bullet}$, the necklace of size 2, there are 2 possible positions for a new bead. In the new necklace there are now 3 positions to choose from. Inductively, this proves that there are possible $(n-1)!$ construction processes for necklaces with n beads.