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A self-equilibrium Friedman-like urn via stochastic approximation

Shuyang Gao *, Hosam M. Mahmoud

Department of Statistics, The George Washington University, Washington, D.C. 20052, USA

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ABSTRACT

In this study, we propose a class of generalized Friedman urns with random entries. The class has the property of self-equilibrium. We prove the almost-sure convergence to the equilibrium point of this entire class by the method of stochastic approximation. We develop a central limit theorem for the proportion of white balls through the convergence theorem of stochastic approximation algorithms. We then apply this class of schemes to model the effect of trading on the price in the stock market under the efficient market hypothesis.

1. Introduction

Urn models are a type of stochastic processes that evolve by perpetuating drawing samples from an urn containing colored balls and depositing colored balls in the urn, according to what appears in the sample. There is a standard way to capture the rules of the evolution in a *replacement matrix* (more on this below). Recently authors considered rectangular replacement matrices corresponding to multiple ball drawing in each step. Authors resorted to methods, such as martingales (Mahmoud, 2013; Kuba et al., 2013; Kuba and Mahmoud, 2017) and recently stochastic approximation (Renlund, 2010, 2011; Lasmar et al., 2016).

A classic Pólya urn process arises from an urn scheme where we have balls of up to k colors (numbered $1, \ldots, k$) and rules of evolution. At each point of discrete time, a ball is drawn at random. The color of the drawn ball is observed, the ball is placed back in the urn, and rules of ball addition are applied. The rules state that if the ball drawn is of color i, we add $A_{i,j}$ balls of color j, and $A_{i,j}$ is a random variable having support in $\mathbb{Z}$, for $j = 1, \ldots, k$. The drawing is then perpetuated indefinitely. When some of the variables $A_{i,j}$ are negative, tenability issues (long-term survival of the process) arise. In this manuscript, we only consider tenable urns. The dynamics of the process are captured by a $k \times k$ matrix

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \ldots & A_{1,k} \\ A_{2,1} & A_{2,2} & A_{2,3} & \ldots & A_{2,k} \\ \vdots & \vdots & & \ddots & \\ A_{k,1} & A_{k,2} & A_{k,3} & \ldots & A_{k,k} \end{pmatrix}.$$

* Corresponding author.
E-mail addresses: gshuyang@gwu.edu (S. Gao), hosam@gwu.edu (H.M. Mahmoud).

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One such classic urn process is introduced in Friedman (1949). Freidman urn process is a two-color urn scheme ($k = 2$) with a symmetric replacement matrix $\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$, with $\alpha \ge 0$, and $\beta \ge 0$. One momentous feature of this class is the symmetry in the replacement matrix, which is the source that drives its asymptotic behavior. Several special cases in this class have been the subject of study for a long time. These special cases include the case $\alpha = -1$ and $\beta = 1$ (Ehrenfest and Ehrenfest-Afanassjewa, 1907), with replacement matrix $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, and the case $\alpha = 1$ and $\beta = 0$ (Pólya, 1930), with replacement matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Freedman (1965) continues the study of this class and develops an asymptotic theory for Friedman urn. One noticeable characteristic of this class is that the proportion of each color reaches the equilibrium at $\frac{1}{2}$ asymptotically, if $\beta > 0$.

One direction of generalization of Friedman urn is to draw multisets of size s, instead of sampling one ball at each discrete step. Kuba et al. (2013) studies the case in which the replacement matrix follows the opposite reinforcement rule of adding $C(s - w)$ white balls and Cw blue balls, for positive integer C, if the sample contains w white (and consequently $s - w$ blue balls). This work is a generalization of the classic Friedman urn with $\alpha = 0$ and $\beta = C$, for the case of single ball drawing ($s = 1$), i.e., the scheme with replacement matrix $\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$. A central limit theorem for the number of white balls after n draws, W_n, is obtained both via the method of moments and via the martingale convergence theorem in Kuba et al. (2013). Recently, Lasmar et al. (2016) studies a multidrawing multicolor urn, and a central limit theorem is proved by stochastic approximation.

In this note, we focus on a class of balanced two-color urns with multiple drawing and random entries. The mean of the replacement matrix of this class has the opposite reinforcement and symmetric property of the replacement matrix of a Friedman urn. Since the urn is balanced, we add at each step K balls into the urn. Our aim is to study a more general class than the one in Kuba et al. (2013), and to provide extension to a subclass considered in Lasmar et al. (2016). The extension captures additional parametrization introduced by the distributions of the random entries. We call this class of balanced urns growing under multiple drawing and with random entries *self-equilibrium Friedman-like urns*.

The dynamics of an urn scheme with multiple drawing with colors in the set $\mathcal{C}$ are represented by a matrix, the rows of which are indexed by the sets that can appear as samples (of fixed size, say s), and the columns of which are indexed by the colors of the balls in the scheme. We assume a *canonical* representation in which the ith sample contains i blue balls (and consequently, it contains $s - i$ white balls), for $0 \le i \le s$. The entry at row i and column $c \in \mathcal{C}$ of the matrix is the *random* number of balls of color c that we add upon withdrawing the ith set in the canonical representation. This random addition has a distribution on $\{0, 1, \ldots, K\}$. Such a support is called in the sequel the *admissible support*. To keep the balance in the urn, the pair of entries on any row adds up to K. Thus, if one entry is the random variable X, with the above-mentioned support, the other must be $K - X$. We assume this random addition to be independent of the sampling process of the same round given the entire past. In this paper, we are concerned with urns on two colors, we name them white and blue.

Definition 1.1 (*Self-equilibrium Friedman-like Urn Schemes*)**.** Suppose that, in a balanced urn on white and blue balls, we draw a sample of odd size s comprised of w white balls and b blue balls. We replace the sample in the urn and execute a prespecified replacement matrix. Such an urn scheme is referred to as a **self-equilibrium Friedman-like** urn scheme,[1] if the following criteria hold[2] :

(1) If $w > b$, we add random numbers of white and blue balls (from the admissible support), such that the mean of the distribution of the blue addition is larger than that of the white addition;
(2) If $w < b$, we add random numbers of white and blue balls, such that the mean of the distribution of the white addition is larger than that of the blue addition;
(3) The addition is symmetric, in the sense that, if for $w = i$ and $b = s - i$, we add X white balls and $K - X$ blue balls, for some random variable X in the admissible support, then for $w = s - i$ and $b = i$, we add $K - Y$ white balls and Y blue balls, where Y is a random variable with the admissible support and with the same mean as X.[3]

Combining the opposite reinforcement rules (1) and (2) in Definition 1.1 with the pairing rule of symmetry (3), the replacement matrix is given by

$$\mathbf{A_n} = \begin{pmatrix} X_{n,0} & Y_{n,0} \\ X_{n,1} & Y_{n,1} \\ \vdots & \vdots \\ X_{n,\frac{s-1}{2}} & Y_{n,\frac{s-1}{2}} \\ X_{n,\frac{s+1}{2}} & Y_{n,\frac{s+1}{2}} \\ \vdots & \vdots \\ X_{n,s-1} & Y_{n,s-1} \\ X_{n,s} & Y_{n,s} \end{pmatrix},$$

1 For brevity, henceforth we refer to the scheme by the name self-equilibrium urn.

2 If s is even a tie ($w = b$) is possible; in this case we can extend the class by imposing a rule by which we add random numbers of white and blue balls, such that the expectation of each is $\frac{K}{2}$. For the time being this case is excluded as our goal is to deal with bona fide opposite reinforcement.

3 The random entries on the different rows may be dependent.