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Stochastic differential equations driven by fractional Brownian motion

Liping Xu [a,*], Jiaowan Luo [b]

[a] School of Information and Mathematics, Yangtze University, Jingzhou, Hubei 434023, China
[b] School of Mathematics and Information Sciences,Guangzhou University, Guangzhou 510006, PR China

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ABSTRACT

In this paper, we are concerned with a class of stochastic differential equations driven by fractional Brownian motion with Hurst parameter $1/2 < H < 1$. By approximation arguments and a comparison theorem, we prove the existence of solutions to this kind of equations driven by fractional Brownian motion under the linear growth condition. Subsequently, by employing Skorokhod's selection theorem, we study the variation of solution to this kind of equations driven by fractional Brownian motion with respect to the initial data.

1. Introduction

Consider the following stochastic differential equation:

$$X_t = X_0 + \int_0^t \sigma(s, X_s)dB_s^H + \int_0^t b(X_s)ds \quad t \in [0, T], \tag{1.1}$$

where $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ are two Borel functions, the integral $\int_0^t \sigma(s, X_s)dB_s^H$ is pathwise Riemann–Stieltjes integral, and $B^H = \{B_t^H, t \in [0, T]\}$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. That is, B^H is a centered Gaussian process with covariance

$$R_H(t, s) = \mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

In the case $H = \frac{1}{2}$, the process B^H is the standard Brownian motion and the existence of a weak solution to (1.1) is well-known by the results of Zvonkin (1974) and Veretennikov (1981), assuming only that the coefficient $b(x)$ satisfies the following linear growth in x

$$|b(x)| \le C(1 + |x|). \tag{1.2}$$

In the singular case $H < \frac{1}{2}$, Nualart and Ouknine (2002) established the existence of a strong solution to (1.1) with $\sigma \equiv 1$ by applying the Girsanov theorem, also assuming only that the coefficient $b(t, x)$ has linear growth in x uniformly in time t. In the regular case $H > \frac{1}{2}$, Boufoussi and Ouknine (2003) proved that (1.1) with $\sigma \equiv 1$ has a unique strong solution if the coefficient b satisfies the linear growth condition (1.2) and is a nonincreasing and continuous function.

* Corresponding author.
E-mail address: xlp211@126.com (L. Xu).

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In Nualart and Răşcanu (2002), the authors established the existence and uniqueness of solution to (1.1) when the drift coefficient b was local Lipschitz continuity and the linear growth and σ satisfied some suitable conditions. The main motivation of our work is to seek an answer to the following interesting question: when $H > \frac{1}{2}$, is there a strong solution to (1.1), assuming only that the coefficient $b(x)$ has linear growth in x? In this paper, we will obtain the existence of the strong solution to (1.1) with Hurst parameter $H > 1/2$ under the condition (1.2) by approximation arguments and a comparison theorem. Besides, the continuous dependence of the solution to (1.1) with respect to the initial condition has been established in Bahlali et al. (1998). But, to the best of our knowledge, there are few results on the continuous dependence of the solution to stochastic differential equations driven by fractional Brownian motion with respect to the initial condition when the fractional Brownian motion is replaced by the standard Brownian motion. We mention that the continuous dependence of the solution to (1.1) with respect to the initial condition established in El Barrimi and Ouknine (2016) in case of $0 < H < \frac{1}{4}$ and $\sigma \equiv 1$. To this end, the second purpose of this paper is to discuss the continuous dependence of the solution to (1.1) with respect to the initial condition when $\frac{1}{2} < H < 1$ and σ is not dependent on X.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. In Section 3, we present and prove our main results.

2. Preliminaries

Let $\frac{1}{2} < H < 1$, $1 - H < \alpha < \frac{1}{2}$. Denote by $W_0^{\alpha,\infty}([0,T];\mathbb{R})$ the space of measurable functions $f : [0,T] \to \mathbb{R}$ such that

$$\|f\|_{\alpha,\infty} := \sup_{t\in[0,T]} \left(|f(t)| + \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}} ds \right) < \infty,$$

and for any $\lambda \geq 0$ an equivalent norm is defined by

$$\|f\|_{\alpha,\infty} := \sup_{t\in[0,T]} e^{-\lambda t} \left(|f(t)| + \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}} ds \right) < \infty.$$

For any $0 < \lambda \leq 1$, denote by $C^\lambda([0,T];\mathbb{R})$ the space of λ-Hölder continuous functions $f : [0,T] \to \mathbb{R}$, equipped with the norm

$$\|f\|_\lambda := \|f\|_\infty + \sup_{0\leq s<t\leq T} \frac{|f(t)-f(s)|}{(t-s)^\lambda} < \infty,$$

where $\|f\|_\infty := \sup_{t\in[0,T]} |f(t)|$. We have, for all $0 < \varepsilon < \alpha$

$$C^{\alpha+\varepsilon}([0,T];\mathbb{R}) \subset W_0^{\alpha,\infty}([0,T];\mathbb{R}) \subset C^{\alpha-\varepsilon}([0,T];\mathbb{R}).$$

Fix a parameter $0 < \alpha < \frac{1}{2}$. Denote by $W_T^{1-\alpha,\infty}([0,T];\mathbb{R})$ the space of measurable functions $g : [0,T] \to \mathbb{R}$ such that

$$\|g\|_{1-\alpha,\infty,T} := \sup_{0\leq s<t\leq T} \left(\frac{|g(t)-g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y)-g(s)|}{(y-s)^{2-\alpha}} dy \right) < \infty.$$

Clearly,

$$C^{1-\alpha+\varepsilon}([0,T];\mathbb{R}) \subset W_T^{1-\alpha,\infty}([0,T];\mathbb{R}) \subset C^{1-\alpha}([0,T];\mathbb{R}), \quad \forall \varepsilon > 0.$$

Now, let us consider the following assumptions on the coefficients, which are supposed to hold for $\mathbb{P}$-almost all $\omega \in \Omega$. The constants M_N, L_N and L_0 may depend on ω.

(H1) $\sigma(t,x)$ is differentiable in x, and there exist some constants $M_0 > 0$, $0 < \beta, \delta \leq 1$ and for every $N \geq 0$ there exists $M_N > 0$ such that the following properties hold:

$$(H_\sigma): \begin{cases} (i) \text{ Lipschitz continuity} \\ \quad |\sigma(t,x) - \sigma(t,y)| \leq M_0|x-y|, \quad \forall x, y \in \mathbb{R}, \ \forall t \in [0,T], \\ (ii) \text{ Local Hölder continuity} \\ \quad |\partial_x\sigma(t,x) - \partial_y\sigma(t,y)| \leq M_N|x-y|^\delta, \quad \forall |x|, |y| \leq N, \ \forall t \in [0,T], \\ (iii) \text{ Hölder continuity in time} \\ \quad |\sigma(t,x) - \sigma(s,x)| + |\partial_x\sigma(t,x) - \partial_y\sigma(s,x)| \leq M_0|t-s|^\beta, \ \forall x \in \mathbb{R}, \ \forall t, s \in [0,T]. \end{cases}$$

(H2) There exists a constant $L_0 > 0$ such that the following properties hold:

$$(H_b): \begin{cases} (i) \ b(\cdot) \text{ is a continuous function,} \\ (ii) \text{ Linear growth} \\ \quad |b(x)| \leq L_0(1+|x|), \quad \forall x \in \mathbb{R}. \end{cases}$$