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On the partial identification of a new causal measure for ordinal outcomes

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ABSTRACT

In a recent paper, Chiba (2017) proposed a new causal measure for ordinal outcomes. We derive the sharp bounds of Chiba (2017)'s causal measure, assuming fixed marginal distributions of the potential outcomes. We illustrate our results via a numeric example.

1. Introduction

Recent years have witnessed a growing interest within the causal inference community to develop methodologies to evaluate causal effects on ordinal outcomes, which are very common in scientific research (Agresti, 2010). One key challenge associated with ordinal outcomes is that the concept of "average" is usually not well-defined. For example, in educational studies it is difficult to "average" bachelor and master degrees. In clinical trials, it is usually unclear what the "mid-point" between mild and severe symptoms is.

To address this problem, numerous alternative causal measures are proposed (e.g., Cheng, 2009; Agresti, 2010; Díaz et al., 2016; Agresti and Kateri, 2017). In a recent paper, Chiba (2017) proposed a new causal measure for ordinal outcomes, by modifying a causal measure previously advocated by Lu et al. (2016). Because we cannot jointly observe the treatment and control potential outcomes, the proposed causal measure is unidentifiable, even for randomized experiments. To circumvent this issue, Chiba (2017) developed a numeric procedure to bound the causal measure. In this note, we provide an alternative perspective, by studying the partial identification (Richardson et al., 2014) of Chiba (2017)'s causal measure, under the assumption of fixed marginal distributions of the potential outcomes. We derive closed-form expressions of the sharp bounds of Chiba (2017)'s new causal measure.

The remainder of this note is organized as follows. Section 2 briefly reviews the potential outcome (Neyman, 1923; Rubin, 1974) based causal inference framework, focusing on ordinal outcomes. Section 3 adopts the partial identification philosophy and derives the closed-form expressions of Chiba (2017)'s new causal measure for ordinal outcomes, and illustrates our results via a numeric example. Section 4 concludes and discusses future directions.

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2. Causal inference for ordinal outcomes

2.1. Potential outcomes

Following the existing literature (Lu et al., 2016; Chiba, 2017), we consider a study with N units, a binary treatment, and an ordinal outcome with J categories labeled as $0, \ldots, J-1$, where 0 and $J-1$ represent the worst and best categories, respectively. Under the Stable Unit Treatment Value Assumption (Rubin, 1980), we define the pair $\{Y_i(1), Y_i(0)\}$ as the potential outcomes of the ith unit under treatment and control, respectively.

For all $k, l = 0, \ldots, J-1$, let $p_{kl} = \text{pr}\{Y_i(1) = k, Y_i(0) = l\}$ denote the probability of units whose potential outcome is k under treatment and l under control. We denote the row and column sums of the probability matrix $\boldsymbol{P} = (p_{kl})_{0 \le k, l \le J-1}$ by

$$p_{k+} = \sum_{l'=0}^{J-1} p_{kl'}, \quad p_{+l} = \sum_{k'=0}^{J-1} p_{k'l} \quad (k, l = 0, 1, \ldots, J-1),$$

which are the marginal distributions of the treatment and control potential outcomes.

2.2. Causal measures for ordinal outcomes

For ordinal outcomes, because the classic average treatment effect $N^{-1}\sum_{i=1}^{N}\{Y_i(1) - Y_i(0)\}$ is not well-defined, we need to consider alternative causal measures, for example the distributional causal effects (cf. Ju and Geng, 2010)

$$\Delta_j = \text{pr}\{Y_i(1) \ge j\} - \text{pr}\{Y_i(0) \ge j\} = \sum_{k \ge j} p_{k+} - \sum_{l \ge j} p_{+l} \quad (j = 0, \ldots, J-1). \tag{1}$$

Echoing several biomedical (e.g., Zhou, 2008; Huang et al., 2017) and social (e.g., Djebbari and Smith, 2008) science researchers, Lu et al. (2016) advocated to measure the probability that the treatment does not harm the experimental units:

$$\tau = \text{pr}\{Y_i(1) \ge Y_i(0)\} = \sum_{k=0}^{J-1} \sum_{l=0}^{k} p_{kl}.$$

Realizing that we enjoy better interpretability after subtracting the term $\text{pr}\{Y_i(1) = 0, Y_i(0) = 0\}$ from τ, in a recent paper (Chiba, 2017) proposed

$$\theta = \sum_{k=1}^{J-1} \sum_{l=0}^{k} p_{kl} = \tau - \text{pr}\{Y_i(1) = 0, Y_i(0) = 0\}.$$

In particular, for binary outcomes (i.e., $J = 2$) the new causal measure θ reduces to the classic causal risk $\text{pr}\{Y_i(1) = 1\}$.

3. Partial identification of causal measures

3.1. Background

For any experimental unit, we cannot jointly observe the treatment and control potential outcomes, rendering the joint probabilities p_{kl}'s, and consequently the causal measures τ and θ, unidentifiable from any observed data. However, fortunately, the marginal probabilities p_{k+}'s and p_{+l}'s may be identifiable (e.g., in randomized experiments). This observation leads to the following assumption, which we employ throughout this note.

Assumption 1. The marginal distributions of the potential outcomes p_{k+}'s and p_{+l}'s are fixed non-negative constants such that $\sum_{k=0}^{J-1} p_{k+} = 1$ and $\sum_{l=0}^{J-1} p_{+l} = 1$.

Assumption 1 implies the possibility to "partially" identify the causal measures. To be more specific, we can derive the sharp lower and upper bounds of a specific causal measure, which are defined as its minimal and maximal values under the following constraints:

$$\sum_{l'=0}^{J-1} p_{kl'} = p_{k+}, \quad \sum_{k'=0}^{J-1} p_{k'l} = p_{+l}, \quad p_{kl} \ge 0 \quad (k, l = 0, \ldots, J-1). \tag{2}$$

Inspired by the existing literature (e.g., Frank et al., 1987) on sharply bounding the distribution of the difference of continuous potential outcomes $Y_i(1) - Y_i(0)$, Lu et al. (2016) derived the sharp bounds of τ, for ordinal outcomes. We (heavily) rely on the following lemma, which generalizes the discussions by Strassen (1965).