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# On asymptotics related to classical inference in stochastic differential equations with random effects

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ABSTRACT

Delattre et al. (2013) considered $n$ independent stochastic differential equations (*SDE*'s), where in each case the drift term is associated with a random effect, the distribution of which depends upon unknown parameters. Assuming the independent and identical (*iid*) situation the authors provide independent proofs of weak consistency and asymptotic normality of the maximum likelihood estimators (*MLE*'s) of the hyper-parameters of their random effects parameters.

In this article, as an alternative route to proving consistency and asymptotic normality in the *SDE* set-up involving random effects, we verify the regularity conditions required by existing relevant theorems. In particular, this approach allowed us to prove strong consistency under weaker assumption. But much more importantly, we further consider the independent, but non-identical set-up associated with the random effects based *SDE* framework, and prove asymptotic results associated with the *MLE*'s.

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## 1. Introduction

Delattre et al. (2013) study mixed-effects stochastic differential equations (*SDE*'s) of the following form:

$$dX_i(t) = b(X_i(t), \phi_i)dt + \sigma(X_i(t))dW_i(t), \quad \text{with } X_i(0) = x^i,\ i = 1, \dots, n. \tag{1.1}$$

Here, for $i = 1, \dots, n$, the stochastic process $X_i(t)$ is assumed to be continuously observed on the time interval $[0, T_i]$ with $T_i > 0$ known, and $\{x^i;\ i = 1, \dots, n\}$ are the known initial values of the $i$th process. The processes $\{W_i(\cdot);\ i = 1, \dots, n\}$ are independent standard Brownian motions, and $\{\phi_i;\ i = 1, \dots, n\}$ are independently and identically distributed (*iid*) random variables with common distribution $g(\varphi, \theta)d\nu(\varphi)$ (for all $\theta$, $g(\varphi, \theta)$ is a density with respect to a dominating measure on $\mathbb{R}^d$, where $\mathbb{R}$ is the real line and $d$ is the dimension), which are independent of the Brownian motions. Here $\theta \in \Omega \subset \mathbb{R}^p$ ($p \geq 2d$) is an unknown parameter to be estimated. The functions $b : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ and $\sigma : \mathbb{R} \mapsto \mathbb{R}$ are the drift function and the diffusion coefficient, respectively, both assumed to be known. Delattre et al. (2013) impose regularity conditions that ensure existence of solutions of (1.1). We adopt their assumptions, which are as follows.

(H1) (i) The function $(x, \varphi) \mapsto b(x, \varphi)$ is $C^1$ (differentiable with continuous first derivative) on $\mathbb{R} \times \mathbb{R}^d$, and such that there exists $K > 0$ so that

$$b^2(x, \varphi) \leq K(1 + x^2 + |\varphi|^2),$$

for all $(x, \varphi) \in \mathbb{R} \times \mathbb{R}^d$.

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(ii) The function $\sigma(\cdot)$ is $C^1$ on $\mathbb{R}$ and
$$\sigma^2(x) \le K(1+x^2),$$
for all $x \in \mathbb{R}$.

(H2) Let $X_i^{\varphi}$ be associated with the SDE of the form (1.1) with drift function $b(x,\varphi)$. Also letting $Q_{\varphi}^{x^i,T_i}$ denote the joint distribution of $\left\{X_i^{\varphi}(t);\ t\in[0,T_i]\right\}$, it is assumed that for $i=1,\ldots,n$, and for all $\varphi,\varphi'$, the following holds:
$$Q_{\varphi}^{x^i,T_i}\left(\int_0^{T_i}\frac{b^2\left(X_i^{\varphi}(t),\varphi'\right)}{\sigma^2(X_i^{\varphi}(t))}dt<\infty\right)=1.$$

(H3) For $f=\frac{\partial b}{\partial\varphi_j}$, $j=1,\ldots,d$, there exist $c>0$ and some $\gamma\ge 0$ such that
$$\sup_{\varphi\in\mathbb{R}^d}\frac{|f(x,\varphi)|}{\sigma^2(x)}\le c\left(1+|x|^{\gamma}\right).$$

Statistically, the $i$th process $X_i(\cdot)$ can be thought of as modelling the $i$th individual and the corresponding random variable $\phi_i$ denotes the random effect of individual $i$. For statistical inference, we follow Delattre et al. (2013) who consider the special case where $b(x,\phi_i)=\phi_i b(x)$. We assume

(H1′) (i) $b(\cdot)$ and $\sigma(\cdot)$ are $C^1$ on $\mathbb{R}$ satisfying $b^2(x)\le K(1+x^2)$ and $\sigma^2(x)\le K(1+x^2)$ for all $x\in\mathbb{R}$, for some $K>0$.
(ii) Almost surely for each $i\ge 1$,
$$\int_0^{T_i}\frac{b^2(X_i(s))}{\sigma^2(X_i(s))}ds<\infty.$$

Under this assumption, (H3) is no longer required; see Delattre et al. (2013). Moreover, Proposition 1 of Delattre et al. (2013) holds; in particular, if for $k\ge 1$, $E|\phi_i|^{2k}<\infty$, then for all $T>0$,
$$\sup_{t\in[0,T]} E\left[X_i(t)\right]^{2k}<\infty. \tag{1.2}$$

As in Delattre et al. (2013) we assume that $\phi_i$ are normally distributed. Hence, (1.2) is satisfied in our case. Delattre et al. (2013) show that the likelihood, depending upon $\theta$, admits a relatively simple form composed of the following sufficient statistics:
$$U_i=\int_0^{T_i}\frac{b(X_i(s))}{\sigma^2(X_i(s))}dX_i(s),\qquad V_i=\int_0^{T_i}\frac{b^2(X_i(s))}{\sigma^2(X_i(s))}ds,\quad i=1,\ldots,n. \tag{1.3}$$

The exact likelihood is given by
$$L(\theta)=\prod_{i=1}^{n}\lambda_i(X_i,\theta), \tag{1.4}$$

where
$$\lambda_i(X_i,\theta)=\int_{\mathbb{R}}g(\varphi,\theta)\exp\left(\varphi U_i-\frac{\varphi^2}{2}V_i\right)d\nu(\varphi). \tag{1.5}$$

Assuming that $g(\varphi,\theta)d\nu(\varphi)\equiv N\left(\mu,\omega^2\right)$, Delattre et al. (2013) obtain the following form of $\lambda_i(X_i,\theta)$:
$$\lambda_i(X_i,\theta)=\frac{1}{\left(1+\omega^2V_i\right)^{1/2}}\exp\left[-\frac{V_i}{2\left(1+\omega^2V_i\right)}\left(\mu-\frac{U_i}{V_i}\right)^2\right]\exp\left(\frac{U_i^2}{2V_i}\right), \tag{1.6}$$

where $\theta=(\mu,\omega^2)\in\Omega\subset\mathbb{R}\times\mathbb{R}^+$. As in Delattre et al. (2013), here we assume that

(H2′) $\Omega$ is compact.

Delattre et al. (2013) consider $x^i=x$ and $T_i=T$ for $i=1,\ldots,n$, so that the set-up boils down to the *iid* situation, and investigate asymptotic properties of the *MLE* of $\theta$, providing proofs of consistency and asymptotic normality independently, without invoking the general results already existing in the literature. In this article, as an alternative, we prove asymptotic properties of the *MLE* in this *SDE* set-up by verifying the regularity conditions of relevant theorems already existing in the literature. Our approach allowed us to prove strong consistency of *MLE*, rather than weak consistency proved by Delattre et al. (2013). Also, importantly, our approach does not require Assumption (H4) of Delattre et al. (2013) which required $(U_1,V_1)$ to have density with respect to the Lebsegue measure on $\mathbb{R}\times\mathbb{R}^+$, which must be jointly continuous and positive on an open ball of $\mathbb{R}\times\mathbb{R}^+$.

Far more importantly, we consider the independent but non-identical case (we refer to the latter as non-*iid*), and prove consistency and asymptotic normality of the *MLE* in this set-up. In what follows, in Section 2 we investigate asymptotic

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