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# Karamata functions and differential equations: Achievements from the 20th century

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**Abstract**

This paper presents the major achievements of the 20th century regarding Karamata functions and the theory of differential equations, made mostly by V. Marić, M. Tomić, E. Omey, J.L. Geluk. The connection between these notions was first noticed by V.G. Avakumović (1910–1990). Slowly and regularly varying functions were introduced by J. Karamata (1902–1967). A group of mathematicians from the Karamata School of classical mathematical analysis were pioneers in research on these functions and their role in the theory of differential equations. Special attentions is given to the study of the Thomas–Fermi, Emden–Fowler and Friedmann equations, as well as the classical second order linear differential equations.


**Sažetak**

U radu su prikazani glavni rezultati istraživanja veze Karamatinih funkcija i teorije diferencijalnih jednačina koji su nastali u 20. veku i čiji su najznačajniji autori bili: V. Marić, M. Tomić, E. Omey, J.L. Geluk. Vezu ove dve oblasti matematičke analize prvi je uočio V.G. Avakumović (1910–1990). Pravilnopromenljive funkcije definisao je i njihova bitna svojstva dao J. Karamata (1902–1967). Grupa matematičara iz Karamatine škole klasične matematičke analize prvi su proučavali te funkcije i njihovu ulogu u teoriji diferencijalnih jednačina. Posebno su značajne Tomas–Fermijeva, Emden–Faulerova i Fridmanova diferencijalne jednačine, kao i klasična linearna diferencijalna jednačina drugog reda.


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*Keywords:* Karamata functions; Regular variation; Differential equations; Thomas–Fermi equations; V.G. Avakumović; V. Marić; M. Tomić

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## 1. Introduction

The term "regularly varying functions" holds a special place in the history of mathematics in Serbia, considering that it was Jovan Karamata (1902–1967), the most distinguished and internationally renowned Serbian mathematician from the first half of the 20th century, who introduced this new class of functions, the regularly varying functions, which were later called Karamata functions. Karamata was the first to precisely define these functions, as well as discover their structure and their fundamental properties. A group of mathematicians from his school, known as the Karamata School of classical mathematical analysis, consisting of Vojislav G. Avakumović (1910–1990), Slobodan Aljančić (1922–1993), Ranko Bojanić (1924–2017), Miodrag Tomić (1912–2001) and Vojislav Marić (1930) were pioneers in research on regular variation, especially its role in the theory of differential equations.[1]

Due to a large number of publications on the Karamata theory and its applications in differential equations, presenting a comprehensive account on the topic is a very challenging task. Thus, the main goal of this paper is to present an overview of the most influential achievements, focusing on those from the 20th century. There are three primary subjects which establish a connection between the Karamata theory and the theory of differential equations. They are: the existence of regularly, especially slowly varying solutions; estimates of such solutions; and their asymptotic behavior. Serbian mathematician V.G. Avakumović noticed and shed light on this connection in three of his papers, written just after World War II. In the following years, Karamata's theory found its applications in partial and functional differential equations, difference equations and systems of differential equations, establishing the terms regularly varying functions and regularly varying sequences as important concepts in these areas. In the first half of the past century, special attention was given to differential equations of mathematical physics – the Thomas–Fermi (nonlinear) equation – and astrophysics and cosmology – the Emden–Fowler (nonlinear) and Friedmann (linear) equations. The mathematical aspect of these subjects was covered in detail in two notable monographs. The first one [Marić, 2000], which could be considered the turning point in connecting the two fields of mathematical analysis in modern times and contains results up to the year 2000, and the second [Řehák, 2014], which contains more recent results.

First, we will give a short survey of the roots of regular variation and Karamata functions, a unique discovery made by Serbian mathematicians of lasting international importance.

### *1.1. Roots of regular variation*

During the 19th century some mathematicians – Niels Henrik Abel (1802–1829), Otto Hölder (1859–1937), Ernesto Cesàro (1859–1906), Émile Borel (1871–1956) – who dealt with classical analysis, the branch of mathematics which no longer holds the same significance it used to, defined the more general notion of the sum of the series by various procedures, later called summability methods under their names, and thus made such otherwise divergent series summable.[2] From the convergence of the series follow these newly introduced summability methods, but the reverse was not always true. In the year 1897, a Slovakian

[1] A similar, but fundamentally different term – the Yugoslav School of Mathematics – was first used in two well-known monographs on regular variation [Seneta, 1976] and [Bingham et al., 1987] and it was primarily used to refer to Karamata's students and associates: V.G. Avakumović, M. Tomić, Ranko Bojanić, Slobodan Aljančić, Vladeta Vučković (1923–2012) and the somewhat younger Časlav Stanojević (1928–2008), Bogdan Bajšanski (1930–2008), Dušan Adamović (1928–2008), V. Marić, Dragoljub Arandjelović (1942–2010) and Zvezdana Radašin (1945–1993). The youngest generation includes Tatjana Ostrogorski (1950–2005), Slobodanka Janković (1951), Slavko Simić (1950), Jelena V. Manojlović (1969) and Jelena Milošević (1979). Five mathematicians affiliated with this school became members of the Serbian Academy of Arts and Sciences, and four of them professors at prominent universities in the U.S.

[2] About divergent series see [Hardy, 1949].

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