Systems & Control Letters 87 (2016) 1–9

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

An optimal control approach to the optical flow problem

Viorel Barbu [a,*], Gabriela Marinoschi [b]

[a] Al.I. Cuza University of Iasi, Octav Mayer Institute of Mathematics, Romanian Academy, 8 Bdul. Carol I, Iasi, Romania

[b] Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Calea 13 Septembrie 13, Bucharest, Romania

ARTICLE INFO

Article history:
Received 15 January 2015
Received in revised form
6 October 2015
Accepted 17 October 2015
Available online 14 November 2015

Keywords:
Optical flow
Gaussian process
Stochastic differential equations
Optimal control

ABSTRACT

The optical flow problem is reduced to an optimal control problem governed by a linear parabolic equation having the unknown velocity field (the optical flow) as drift term. This model is derived from a new assumption, that is, the brightness intensity is conserved on a moving pattern driven by a Gaussian stochastic process. The optimality conditions are deduced by a passage to the limit technique in an approximating optimal control problem introduced for a regularization purpose. Finally, the controller uniqueness is addressed.

1. Introduction

The optical flow problem consists in determining the motion, or more exactly the velocity field, of an object function representing the brightness pattern in an image. Let us briefly present the standard approach of this problem based on the linear transport equations (see, e.g., [1–5]).

Consider a continuous sequence of images $I = I(t,x)$, $t \in [0,T]$, defined on a domain $\mathcal{O} \subset \mathbb{R}^d$, $d = 1, 2, 3$. More precisely, $I(t,x)$ is the brightness of the pattern at point $x = (x_1, x_2, \ldots, x_d)$ at time t. Denote by $X = X(t) = (X_1(t), X_2(t), \ldots, X_d(t))$ the trajectory of the moving pattern defined by the system of differential equations

$$\frac{dX(t)}{dt} = U(t, X(t)), \quad 0 \le t \le T, \tag{1}$$

where $U : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the velocity field of the motion.

Assuming that the brightness is constant in time along the trajectory (this is the basic hypothesis of the theory), that is

$$\frac{d}{dt} I(t, X(t)) = 0, \quad \text{for all } t \in [0,T], \tag{2}$$

we are led to the linear transport equation

$$\frac{\partial}{\partial t} I(t,x) + U(t,x) \cdot \nabla I(t,x) = 0, \quad t \in [0,T],\ x \in \mathcal{O}. \tag{3}$$

Here, ∇ denotes the gradient operator.

The computation of the optical flow U as a function of the brightness intensity and its derivatives $\frac{\partial I}{\partial t}$, ∇I, in given sequences of images, is a fundamental problem of the artificial vision. As largely documented in the literature, (3) is not sufficient to determine the velocity field U, so that other constraints derived from variational principle are necessary (see [6,7]).

In this work we propose a different approach based on the constant brightness assumption (2), but which leads to a stochastic linear parabolic equation for I, with the drift term U. In fact, if the motion of the pattern is not deterministic but, as usually happens in the real world, it is driven by an additional Gaussian process W, Eq. (1) must be replaced by the Itô stochastic differential equation

$$dX(t) = U(t, X(t))dt + dW(t), \quad 0 \le t \le T, \tag{4}$$
$$X(0) = x,$$

where $W(t)$ is a Gaussian (Wiener) process of the form $W(t) = \left\{ \sum_{i=1}^d a_{ij}\beta_i(t) \right\}_{j=1}^d$. Here, $\{\beta_i\}_{i=1}^d$ is a system of independent Brownian motions in a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$, with the natural filtration $(\mathcal{F}_t)_{t\ge 0}$, and the real matrix $\left(a_{ij}\right)_{i,j=1}^d$ is not singular.

Then, the brightness conservation law

$$dI(t, X(t)) = 0 \tag{5}$$

leads, via Itô's formula (see e.g., [8]), to the equation

$$\frac{\partial}{\partial t} I(t, X(t)) + U(t, X(t)) \cdot \nabla I(t, X(t))dt + \frac{1}{2} \sum_{i,j=1}^d b_{ij} \frac{\partial^2 I}{\partial X_i \partial X_j}(t, X(t))dt + \nabla I(t, X(t)) \cdot dW(t) = 0, \tag{6}$$

along every path X, where $b_{ij} = \sum_{k=1}^d a_{ik}a_{kj}$, $i, j = 1, \ldots, d$.

* Corresponding author.
E-mail addresses: vb41@uaic.ro (V. Barbu), gabriela.marinoschi@acad.ro (G. Marinoschi).

http://dx.doi.org/10.1016/j.sysconle.2015.10.004
0167-6911/

We set $J(t,x)=E[I(X(t,x))]$, where E is the expectation in the probability space $\{\Omega,\mathcal{F},\mathbb{P}\}$, and get by (6) the equation

$$\frac{\partial}{\partial t}J(t,x)+U(t,x)\cdot\nabla J(t,x)+\frac{1}{2}L(J(t,x))=0, \tag{7}$$

$\forall x\in\mathbb{R}^d,\ t\in(0,T)$. Here L is the second order elliptic operator

$$L(y)=\sum_{i,j=1}^{d}b_{ij}\frac{\partial^2 y}{\partial x_i\partial x_j},\quad y\in C^2(\mathbb{R}^d), \tag{8}$$

(see, e.g., [8, p. 68]).
A final Cauchy condition of the form $J(T,x)=J_1(x)$, $x\in O$, must be imposed. Then the optical flow problem reduces to determining the velocity field $U=U(t,x)$ from the solution J to the backward parabolic equation (7). On a bounded domain $O\subset\mathbb{R}^d$ with smooth boundary ∂O, one adds the Neumann boundary conditions $\frac{\partial}{\partial\nu}J=0$, on $(0,T)\times\partial O$, where $\frac{\partial}{\partial\nu}$ is the outward normal derivative to ∂O. For $b_{ij}=2\mu\delta_{ij}$, $\mu>0$, that is for $L=2\mu\Delta$, this is the classical Neumann boundary condition.

We set in the following $Q:=(0,T)\times O$ and $\Sigma:=(0,T)\times\partial O$.

Our objective in this work is to determine via an optimal control technique the optical flow U which transports an initial image of intensity J_0 to a final image of intensity J_1 in an interval of time T. We shall determine the optical flow $U=U(t,x)$, by the optimal control problem

$$\min\left\{\frac{1}{2}\int_O(J(0,x)-J_0(x))^2dx+\frac{\sigma}{2}\int_0^T\!\!\int_O|U(t,x)|_d^2\,dxdt\right\} \tag{9}$$

subject to

$$\begin{aligned}&\frac{\partial J}{\partial t}+\mu\Delta J+U\cdot\nabla J=0\quad\text{in } Q,\\ &J(T,x)=J_1(x)\quad\text{in } O,\qquad \frac{\partial J}{\partial\nu}=0\quad\text{on }\Sigma,\end{aligned} \tag{10}$$

with U such that

$$\nabla\cdot U(t,x)=0,\quad \text{a.e. } (t,x)\in Q, \tag{11}$$

$$U(t,x)\cdot\nu(x)=0,\quad \text{a.e. } (t,x)\in\Sigma. \tag{12}$$

Here σ is an arbitrary real positive number, $\nabla\cdot U=\sum_{i=1}^d\frac{\partial U_i}{\partial x_i}$ and U is the unknown controller. The functions J_0 and J_1 represent known data. In (9), $|\cdot|_d$ denotes the Euclidean norm in $\mathbb{R}^d$.

In other words, we look for the velocity field in the space of free divergence vectors in Q, which are tangential to ∂O. The motivation of the free divergence condition (11) is that the free divergence field U makes the flow volume conserving, smooth and varying not too much inside a non-deforming moving object (see [7]). As regards condition (12), it simply requires that the velocity field is tangential at the boundary, which is quite a natural condition for the pattern dynamics. (In literature, see, e.g., [7], one takes usually the stronger condition $U=0$ on Σ.) We emphasize, however, that problem (9)–(10) is also relevant in other inverse problems involving conservation laws of the form (5) (as, for instance, for retrieving the initial concentration of a pollutant diffused and transported in a fluid from the final observed position and concentration).

Comparing with other literature on optimal flow problems, the works [7,6] should be first mentioned. For instance, the approach [7] is based on an optimal control problem of the form (9) but governed by the linear transport equation

$$\frac{\partial y}{\partial t}-u\cdot\nabla y=0,\quad\text{in }(0,T)\times O,\qquad y(0,x)=y_0(x).$$

This state equation is well-posed in the space $BV(O)$ of functions with bounded variation on Q, but the cost functional Φ is neither coercive, nor lower semicontinuous in $L^2(0,T;BV(O))$. A constraint term $\|u\|^2_{L^2(0,T;(H^3(O))^d)}$ was added in the cost functional in order to have existence. Such a term leads to a high order necessary conditions of optimality.

By comparison, the optimality system we obtain here, which reduces to a forward–backward parabolic system, can be used to set up a gradient based algorithm to compute the optimal flow, but complete numerical tests remain to be done. For numerical analysis of such problems we also refer to [9].

Of course, (10) can be viewed as a parabolic regularization, via the diffusion term $\mu\Delta J$, of the hyperbolic transport equation (3). However, since the state equation (10) is derived from the stochastic brightness conservation hypothesis (5), we see (10)–(12) not as a simplification of the classical optical flow model, but as a natural setting for this problem.

Notations. Everywhere in the following, O is a bounded and open domain of $\mathbb{R}^d$, $d=1,2,3$, with smooth boundary ∂O. By $L^p(O)$, $1\le p\le\infty$, we denote the space of L^p-Lebesgue integrable functions on O, with the standard norm (denoted by $\|\cdot\|_{L^p(O)}$). By $W^{k,p}(O)$, $k>0$, $1\le p\le\infty$, we denote the standard Sobolev spaces on O (see e.g. [10]). For $p=2$ we set $W^{k,2}(O)=H^k(O)$ and denote by $(H^1(O))'$ the dual space of $H^1(O)$. If Y is a Banach space we denote by $L^p(0,T;Y)$ the space of all measurable functions $y:(0,T)\to Y$, with $\|y(\cdot)\|_Y\in L^p(0,T)$. (Here, $\|\cdot\|_Y$ is the norm of Y.) By $C([0,T];Y)$ we denote the space of all continuous functions $y:[0,T]\to Y$ and $W^{1,p}([0,T];Y)$ is the space of absolutely continuous functions $y:[0,T]\to Y$ with $\frac{dy}{dt}\in L^p(0,T;Y)$, $1\le p\le\infty$ (see e.g., [11, p. 23]). Finally, let us denote by H the free divergence tangential vectors space (see e.g. [12, p. 7])

$$H=\{u\in(L^2(O))^d;\ \nabla\cdot u=0 \text{ in } O,\ u\cdot\nu=0\text{ on }\partial O\}. \tag{13}$$

We also recall that if $\nabla\cdot u\in L^2(O)$, then the trace $u\cdot\nu$ is well defined, as an element of the distribution space $H^{-1/2}(\partial O)$. H is a closed subspace of $(L^2(O))^d$, and it is a Hilbert space with the scalar product $(u,v)_H=\int_O u(x)\cdot v(x)dx$, and the norm

$$\|u\|_H=\|u\|_{(L^2(O))^d}=\left(\int_O|u(x)|_d^2\,dx\right)^{1/2}. \tag{14}$$

We shall also use the notation $\overline{Q}=[0,T]\times\overline{O}$, where $\overline{O}$ is the closure of O. By $C^\infty(\overline{Q})$, $C^\infty(\overline{O})$ or $C^\infty(\mathbb{R}^d)$ we shall denote the spaces of infinitely differentiable functions on $\overline{Q}$, $\overline{O}$ and $\mathbb{R}^d$.

2. The optimal control problem

Taking $u(t,x)=U(T-t,x)$, $y(t,x)=J(T-t,x)$, $y_0(x)=J_1(x)$ and $y_1(x)=J_0(x)$ we reformulate problem (9) as

$$\begin{aligned}\min_{u\in L^2(0,T;H)}&\left\{\Phi(u)=\frac{1}{2}\int_O(y(T,x)-y_1(x))^2dx\right.\\ &\left.+\frac{\sigma}{2}\int_0^T\int_O|u(t,x)|_d^2\,dxdt\right\},\end{aligned} \tag{P}$$

subject to the forward parabolic equation

$$\begin{aligned}&\frac{\partial y}{\partial t}-\mu\Delta y-u\cdot\nabla y=0\quad\text{in } Q,\\ &y(0,x)=y_0(x),\quad\text{in } O,\qquad \frac{\partial y}{\partial\nu}=0,\quad\text{on }\Sigma.\end{aligned} \tag{15}$$

In the next subsection we introduce the definition of a weak solution to the state system (15) and prove the existence, uniqueness and other properties of this weak solution (Theorem 1). Theorem 2 proves the existence of a solution for the problem (P), i.e., of a minimizer for Φ. In Section 3 an approximating control problem (P_ε), is studied. It requires the minimization of