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Local input-to-state stability: Characterizations and counterexamples

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ABSTRACT

We show that a nonlinear locally uniformly asymptotically stable infinite-dimensional system is automatically locally input-to-state stable (LISS) provided the nonlinearity possesses some sort of uniform continuity with respect to external inputs. Also we prove that LISS is equivalent to existence of a LISS Lyapunov function. We show by means of a counterexample that if this uniformity is not present, then the equivalence of local asymptotic stability and local ISS does not hold anymore. Using a modification of this counterexample we show that in infinite dimensions a uniformly globally asymptotically stable at zero, globally stable and locally ISS system possessing an asymptotic gain property does not have to be ISS (in contrast to finite dimensional case).

1. Introduction

Input-to-state stability (ISS) theory of ordinary differential equations (ODEs) is nowadays a developed theory with a firm theoretical basis, with a variety of powerful tools for investigation of ISS and with a multitude of applications in the nonlinear control theory, in particular to robust stabilization of nonlinear systems [1], design of nonlinear observers, analysis of large-scale networks [2–4], etc.

Among the most important results in ISS theory for ODE systems are the characterizations of ISS in terms of Lyapunov functions and other stability properties [5,6]. These theorems have shown that ISS is a central notion in stability theory of control systems and at the same time these results played an important role in the proofs of other important results, e.g. small-gain theorems in a trajectory formulation [3]. In contrast to global ISS property, according to author's knowledge, there are no characterizations of local ISS property available in the literature. This is quite surprising, since such characterizations are useful from theoretical as well from the practical point of view, and at the same time closely related results for other kinds of robustness are well-known (see e.g. [7, Corollary 4.2.3]). The most known result in ISS context is [6, Lemma 1] telling that global asymptotic stability for a zero input (0-GAS) implies LISS for ODE systems.

One of our aims in this paper is to obtain the characterization of the local ISS property. We do not restrict ourselves to consideration of ODE systems, since the questions of robust stabilization, control and observation of infinite-dimensional systems are of central importance in control theory and we believe that ISS is a right tool to handle these questions. Thus, we study (L)ISS of general infinite-dimensional systems of the form

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in X,\ u(t) \in U, \tag{1}$$

where X is a Banach space, U is a linear normed space, A is the generator of a C_0-semigroup $\{T(t),\ \ t \geq 0\}$ and $f : X \times U \to X$. Many classes of evolution equations, such as parabolic and hyperbolic partial differential equations, can be written in the form (1): [7–9].

In the last years ISS of infinite-dimensional systems (1) as well as of partial differential equations has been studied in a number of papers, see [10–17] to cite a few. But in most of these works the attention has been devoted to construction of Lyapunov functions for ISS systems and to design of robust stabilizing controllers for unstable systems. At the same time the problem of characterizations of local and global ISS for systems (1) is still open.

In this paper we make a step towards its solution. There are two contributions in this paper: a 'positive' and a 'negative' one. Our positive result is that under some sort of uniform continuity of f with respect to external inputs, local uniform asymptotic stability of (1) is equivalent to local ISS of (1) and to existence of a Lipschitz LISS Lyapunov function for it. Thus, our findings imply the result [6, Lemma 1] as a very special case. In the proof of this result we use a technique, used to prove a closely related robustness result for infinite-dimensional systems, see [7, Corollary 4.2.3]. We show by means of a counterexample, that if the nonlinearity f is continuous at the neighborhood of zero, but without additional uniformity, then the main result does not hold.

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Note also that for time-delay systems a related result is available, namely that local exponential stability of the undisturbed systems plus some additional properties of the nonlinearity imply ISS of the disturbed system [18]. However, there are important differences between our paper and [18], in particular, we do not require local exponential stability of (1) and instead we consider merely locally uniformly asymptotically stable systems.

The 'negative' contribution of this paper is an example of a system (obtained by a modification of the previous counterexample), which is locally ISS, uniformly globally asymptotically stable at zero (0-UGAS), globally stable (GS) and possessing an asymptotic gain property (AG), but which is not ISS. This shows that restatements of global ISS property proved in [5,6] for ODE systems (such as ISS = AG + GS = AG + 0-UGAS = AG + LISS), do not hold for systems (1) in general.

We believe that the results obtained in this paper will be useful in applications as well as for the characterization of the global ISS property for abstract systems (1).

2. Preliminaries

Let $\mathbb{R}_+ := [0, \infty)$ and $B_r := \{x \in X : \|x\|_X \leq r\}$. In all the pages below we assume that the set of input values U is a normed linear space with the norm $\|\cdot\|_U$ and that the input functions $u : \mathbb{R}_+ \to U$ belong to the space $\mathcal{U} := PC(\mathbb{R}_+, U)$ of bounded piecewise continuous functions, which are right continuous. The norm of $u \in \mathcal{U}$ we denote as $\|u\|_{\mathcal{U}} := \sup_{t\geq 0} \|u(s)\|_U$.

We are going to study weak solutions of (1), i.e. the solutions of the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(x(s), u(s))ds \quad \forall t \in [0, \tau], \tag{2}$$

belonging to $C([0, \tau], X)$ for some $\tau > 0$.

We assume that $f(0, 0) = 0$, i.e., $x \equiv 0$ is an equilibrium point of (1).

Assumption 1. We suppose throughout the paper that the nonlinearity f satisfies the following properties:

(i) $f : X \times U \to X$ is Lipschitz continuous on bounded subsets of X, uniformly with respect to the second argument, i.e. $\forall C > 0\ \exists L_f(C) > 0$, such that $\forall x, y : \|x\|_X \leq C,\ \|y\|_X \leq C, \forall v \in U$, it holds that

$$\|f(y, v) - f(x, v)\|_X \leq L_f(C)\|y - x\|_X. \tag{3}$$

(ii) $f(x, \cdot)$ is continuous for all $x \in X$.

Since $\mathcal{U} = PC(\mathbb{R}_+, U)$, Assumption 1 ensures that the weak solution of (1) exists and is unique, according to a variation of the classical existence and uniqueness theorem [8, Proposition 4.3.3]. We denote by $\phi(t, x, u)$ this solution at moment $t \in \mathbb{R}_+$ associated with an initial condition $x \in X$ at $t = 0$, and input $u \in \mathcal{U}$.

Also we assume that the solution ϕ depends continuously on initial states and external inputs at the neighborhood of the origin, namely:

Assumption 2. For any $\varepsilon > 0$ and for any $\tau > 0$ there exists $\delta > 0$ so that for any $x \in X : \|x\|_X \leq \delta$ and for any $u \in \mathcal{U} : \|u\|_{\mathcal{U}} \leq \delta$ it follows that $\|\phi(t, x, u)\|_X \leq \varepsilon$, for all $t \in [0, \tau]$.

For the formulation of stability properties we use the comparison functions formalism:

$$\mathcal{K} := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \,|\, \gamma \text{ is continuous, strictly increasing and } \gamma(0) = 0\}$$

$$\mathcal{K}_\infty := \{\gamma \in \mathcal{K} \,|\, \gamma \text{ is unbounded}\}$$

$$\mathcal{L} := \Big\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \,|\, \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\Big\}$$

$$\mathcal{KL} := \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \,|\, \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L},\ \forall t \geq 0,\ \forall r > 0\}.$$

The main notions of this paper are:

Definition 1. System (1) is called

- input-to-state stable (ISS), if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that $\forall x \in X, \forall u \in \mathcal{U}$ and $\forall t \geq 0$ the following holds
$$\|\phi(t, x, u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}). \tag{4}$$
- locally input-to-state stable (LISS), if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $r > 0$ such that the inequality (4) holds $\forall x \in B_r$, $\forall u \in \mathcal{U} : \|u\|_{\mathcal{U}} \leq r$ and $\forall t \geq 0$.

In order to understand the essence of (L)ISS we introduce several other properties:

Definition 2. System (1)

- is globally asymptotically stable at zero uniformly with respect to state (0-UGASs), if $\exists\beta \in \mathcal{KL}$, such that $\forall x \in X, \forall t \geq 0$ the following inequality holds
$$\|\phi(t, x, 0)\|_X \leq \beta(\|x\|_X, t). \tag{5}$$
- is locally asymptotically stable at zero uniformly with respect to state (0-UASs), if for certain $\beta \in \mathcal{KL}$ the estimate (5) holds for all $x \in B_r$ with $r > 0$ small enough.
- is globally stable (GS), if $\exists\sigma \in \mathcal{K}_\infty,\ \gamma \in \mathcal{K}_\infty \cup \{0\}$ such that $\forall x \in X, \forall u \in \mathcal{U}, \forall t \geq 0$ we have
$$\|\phi(t, x, u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}). \tag{6}$$
- has asymptotic gain (AG) property, if $\exists\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $\varepsilon > 0$, for all $x \in X$ and for all $u \in \mathcal{U}$ there exists $\tau_a = \tau_a(\varepsilon, x, u) < \infty$:
$$\forall t \geq \tau_a \quad\Rightarrow\quad \|\phi(t, x, u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \tag{7}$$
- has strong asymptotic gain (sAG) property, if $\exists\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $x \in X$ and for all $\varepsilon > 0$ there exists $\tau_a = \tau_a(\varepsilon, x) < \infty$:
$$\forall t \geq \tau_a, \quad \forall u \in \mathcal{U} \Rightarrow \|\phi(t, x, u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \tag{8}$$

Both AG and sAG imply that all trajectories converge to the ball of radius $\gamma(\|u\|_{\mathcal{U}})$ around the origin as soon as $t \to \infty$. The difference between AG and sAG is in a kind of dependence of τ_a on states and inputs. In sAG systems this time depends on the state x (and may vary for the states with the same norm), but it does not depend on u. In AG systems τ_a depends both on x and on u.

A powerful tool to investigate ISS and LISS of control systems is an ISS/LISS Lyapunov function.

Definition 3. A continuous function $V : D \to \mathbb{R}_+, 0 \in \text{int}(D) \subset X$ is called a LISS Lyapunov function, if there exist $r > 0$ $\psi_1, \psi_2 \in \mathcal{K}_\infty, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that $B_r \subset D$ and

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in B_r \tag{9}$$

and Lie derivative

$$\dot{V}_u(x) := \overline{\lim_{t\to+0}} \frac{1}{t}(V(\phi(t, x, u)) - V(x))$$

of V along the trajectories of the system (1) satisfies

$$\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U) \tag{10}$$

for all $x \in B_r$ and $u \in \mathcal{U} : \|u\|_{\mathcal{U}} \leq r$.