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Research paper

Analytic study of a coupled Kerr-SBS system

Robert Conte[a,b,*], Maria Luz Gandarias[c]

[a] LRC MESO, CEA-DAM-DIF, F-91297 Arpajon, France
[b] Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong
[c] Departamento de Matematicas, Universidad de Cádiz, Casa postale 40, E-11510 Puerto Real, Cádiz, Spain

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ABSTRACT

In order to describe the coupling between the Kerr nonlinearity and the stimulated Brillouin scattering, Mauger et al. recently proposed a system of partial differential equations in three complex amplitudes. We perform here its analytic study by two methods. The first method is to investigate the structure of singularities, in order to possibly find closed form single-valued solutions obeying this structure. The second method is to look at the infinitesimal symmetries of the system in order to build reductions to a lesser number of independent variables. Our overall conclusion is that the structure of singularities is too intricate to obtain closed form solutions by the usual methods. One of our results is the proof of the nonexistence of traveling waves.

1. The coupled Kerr-SBS system

The coupling between Kerr effect and stimulated Brillouin scattering [1] can be described by three complex partial differential equations (PDE) in three complex amplitudes U_1, U_2, Q depending on four independent variables x, y, t, z [8, Eqs. (7)–(9)]

$$\begin{cases} i(U_{1,z}+v_g U_{1,t})+\dfrac{U_{1,xx}+U_{1,yy}}{2k_0}+b\left(|U_1|^2+2|U_2|^2\right)U_1+i\dfrac{g}{2}QU_2=0, \\ -i(U_{2,z}-v_g U_{2,t})+\dfrac{U_{2,xx}+U_{2,yy}}{2k_0}+b\left(|U_2|^2+2|U_1|^2\right)U_2-i\dfrac{g}{2}\overline{Q}U_1=0, \\ \tau Q_t+Q-U_1\overline{U}_2=0, \end{cases} \tag{1}$$

in which v_g, k_0, b, g, τ are real constants. We adopt the notation of nonlinear optics, in which the time t and the longitudinal coordinate z are exchanged as compared to mathematical physics.

* Corresponding author.
E-mail addresses: Robert.Conte@cea.fr (R. Conte), MariaLuz.Gandarias@uca.es (M.L. Gandarias).

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Although we will focus on the generic case $g\tau \neq 0$, we will also consider the two nongeneric cases $g\tau = 0$, for which the system is only four-dimensional,

$$g\tau = 0: \begin{cases} i(U_{1,z} + v_g U_{1,t}) + \dfrac{U_{1,xx} + U_{1,yy}}{2k_0} + \left(b|U_1{}^2 + \left(2b + i\dfrac{g}{2}\right)|U_2{}^2\right)U_1 = 0, \\ -i(U_{2,z} - v_g U_{2,t}) + \dfrac{U_{2,xx} + U_{2,yy}}{2k_0} + \left(b|U_2{}^2 + \left(2b - i\dfrac{g}{2}\right)|U_1{}^2\right)U_2 = 0. \end{cases} \tag{2}$$

At present time, no solution is known to the generic system (1) ($v_g bg\tau \neq 0$). The goal of this work is to look for possible closed form solutions by two methods: singularity analysis, infinitesimal symmetries.

A prerequisite to the search of closed form solutions is to investigate the singularity structure of the system, this is done in Section 2, and this results in a triangular system of five PDEs to be obeyed in order for a closed form solution to exist.

In Section 3, we look for the simplest class of possible closed form solutions, in which U_1, U_2, Q could have shock profiles. We find that, at least for the radial reduction $(U_j, Q) = f(x^2 + y^2, z, t)$, such a solution does not exist.

In Section 4, we apply the classical Lie method, derive the Lie algebra, compute the commutator table and the adjoint table [10].

Finally, in Section 5, we define a few reductions to a lesser number of independent variables.

2. Singularity analysis

There exists only one limiting case in which the system (1) is integrable, this is its degeneracy to the nonlinear Schrödinger equation $U_1 = U_2, g = 0, \partial_z = 0, c_1\partial_x + c_2\partial_y = 0, (c_1, c_2) \neq (0, 0)$. Let us prove that, except for this limiting case, the system (1) is always nonintegrable, in the sense that it always admits a multi-valued behaviour around a singularity which depends on the initial conditions (i.e. what is called a movable singularity). It is convenient to denote the list of dependent variables $(U_1, \overline{U}_1, U_2, \overline{U}_2, Q, \overline{Q})$ as the six-dimensional vector $\mathbf{u}$.

A necessary condition for the system (1) to display a single-valued behaviour of the general solution around any movable singularity (i.e. what is known as the Painlevé property [3]) is that all possible Laurent series locally representing the general solution near a movable singular manifold $\varphi(x, y, z, t) - \varphi_0 = 0$,

$$\mathbf{u} = \sum_{j=0}^{+\infty} \mathbf{u}_j \chi^{j+\mathbf{p}}, \tag{3}$$

indeed exist. In the above series, the expansion variable χ vanishes when $\varphi(x, y, z, t) - \varphi_0 \to 0$. This classical but technical computation [3] generates several necessary conditions, the main ones being the following.

1. At least one of the six components of the leading power $\mathbf{p}$ must not be a positive integer (so that $\chi = 0$ is indeed a singularity).
2. The Fuchs indices of the linearized system near the solution (3) must all be integer (whatever be their sign).
3. For any Fuchs index $j \geq 1$, the (affine) recurrence relation for $\mathbf{u}_j$ must admit a solution, i.e. no logarithms are allowed to enter the expansion, and this requires some conditions (no-logarithm conditions, in short no-log conditions) to be obeyed.

2.1. Generic case $g\tau \neq 0$

There exists a dominant behaviour in which all six complex fields have simple poles (χ is here chosen as $\chi = \varphi(x, y, z, t) - \varphi_0$),

$$\begin{cases} U_1 \sim Me^{ia_1}\chi^{-1}, \ \overline{U}_1 \sim Me^{-ia_1}\chi^{-1}, \ U_2 \sim Me^{ia_2}\chi^{-1}, \ \overline{U}_2 \sim Me^{-ia_2}\chi^{-1}, \\ Q \sim Ne^{ia_1 - ia_2}\chi^{-1}, \ \overline{Q} \sim Ne^{ia_2 - ia_1}\chi^{-1}, \\ M^2 = -\dfrac{\varphi_x^2 + \varphi_y^2}{3k_0 b}, \ N = \dfrac{\varphi_x^2 + \varphi_y^2}{3k_0 b\tau\varphi_t}, \ (\varphi_x^2 + \varphi_y^2)\varphi_t \neq 0. \end{cases} \tag{4}$$

and the two phases a_1, a_2 are arbitrary functions of (x, y, z, t). These two sets of values for the moduli (M, N) define two families of movable singularities. The Fuchs indices of each family are equal to

$$-1, 0, 0, 1, 1, 3, 3, 4, \frac{3}{2} + \frac{\sqrt{11}}{2\sqrt{3}}, \frac{3}{2} - \frac{\sqrt{11}}{2\sqrt{3}}, \tag{5}$$

and the two irrational indices prove the nonintegrability of the system. This however does not yet rule out possible single-valued solutions.

Each of the five indices 1, 1, 3, 3, 4 generates one necessary condition for the Laurent series (3) to exist. If they are all obeyed, the Laurent series depends on the eight arbitrary functions

$$\varphi, a_1, a_2, Q_1, \overline{Q}_1, U_{1,3} - \overline{U}_{1,3}, U_{2,3} - \overline{U}_{2,3}, U_{1,4} + \overline{U}_{1,4} + U_{2,4} + \overline{U}_{2,4}, \tag{6}$$