Commun Nonlinear Sci Numer Simulat 42 (2017) 158–177

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Exact solutions and maximal dimension of invariant subspaces of time fractional coupled nonlinear partial differential equations

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ARTICLE INFO

Article history:
Received 6 August 2015
Revised 16 March 2016
Accepted 10 May 2016
Available online 13 May 2016

Keywords:
Time fractional coupled nonlinear PDEs
Invariant subspace method
Laplace transform method
Mittag-Leffler function

ABSTRACT

We show how invariant subspace method can be extended to time fractional coupled nonlinear partial differential equations and construct their exact solutions. Effectiveness of the method has been illustrated through time fractional Hunter–Saxton equation, time fractional coupled nonlinear diffusion system, time fractional coupled Boussinesq equation and time fractional Whitman–Broer–Kaup system. Also we explain how maximal dimension of the time fractional coupled nonlinear partial differential equations can be estimated.

1. Introduction

In recent years, the study of fractional differential equations has drawn much attention both from the mathematical and physical points of view from researchers in nonlinear phenomena. The idea of fractional order derivative was started with half-order derivative as discussed in the literature by Leibniz and L'Hopital. Next, it was extended to an arbitrary order derivative by Liouville, Riemann, Grunwald, Letnikov, Caputo, etc. In addition, different approaches to define fractional derivatives are known [1–5]. Fractional calculus is also considered as a novel topic [1–3],and it has recently gained popularity and importance. In reality, a natural phenomenon may depend not only the time instant but also the previous time history, which can be successfully modeled by using the theory of derivatives and integrals of fractional order [2,3]. The fractional calculus is a powerful tool to describe physical systems that have long-term memory and long-range spatial interactions [1–6]. Fractional calculus has become of increasing use for analyzing not only stochastic processes driven by fractional Brownian process, but also non-random fractional phenomena in physics, like the study of porous system, for instance and quantum Mechanics [7]. Nonlinear differential equations exhibiting a wide variety of interesting features, such as periodic solution, chaos, strange attractors, fractals, etc., are not exactly solvable in general [8]. Many analytic techniques have been developed to deal with nonlinear differential equations during the recent decades. However, the study of fractional differential equations have been handicapped due to the absence of well-defined analytic techniques to deal with them. The derivation of exact solution of fractional differential equations are not an easy task and it remains a relevant problem.

Several ad hoc methods for solving fractional differential equations were developed recently including Adomian decomposition method [9,10], first integral method [11], homotopy analysis method [12–14], Lie group theory method [15,16],

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http://dx.doi.org/10.1016/j.cnsns.2016.05.017

other semi-analytical techniques [17–24]. Recent investigations show that an analytic method based on invariant subspace provides an effective tool to derive exact solutions of fractional nonlinear partial differential equations. This method was initially proposed by Galaktionov and Svirshchevskii [25] and later on extended by Gazizov and Kasatkin [26] for time fractional nonlinear partial differential equations. Recently, the effectiveness of the invariant subspace method has been demonstrated by [27–30] for time fractional scalar nonlinear partial differential equations(PDEs). It is appropriate to mention that the invariant subspace method has been extended to coupled nonlinear PDEs [25,31–38]. To the best of our knowledge, the invariant subspace method has not been extended to time fractional coupled nonlinear PDEs. The main objective of this article is that how the invariant subspace method can be extended to time fractional coupled nonlinear PDEs and constructs their exact solutions. The advantage of this method is that it is algorithmic and provides an efficient tool to generalize compact soliton solutions and similarity solutions for time fractional nonlinear PDEs. In many cases, the invariant subspace method yields an exact solution of the time fractional coupled nonlinear PDEs expressed in terms of the well known Mittag-Leffler function.

The organization of the article is as follows: For clarity of presentation, in Section 2 we recall some basic definitions and results of fractional derivatives and integrals. Also we provide the salient features of the invariant subspace method applicable to time fractional scalar nonlinear PDEs and show how it can be extended to coupled nonlinear PDEs. In Section 3, we explain how maximal dimension of invariant subspaces admitted by time fractional coupled nonlinear PDEs with different orders can be estimated. The usefulness of the above method has been illustrated through the following: Hunter–Saxton equation [25], coupled nonlinear diffusion system [33], two coupled Boussinesq equation [8] and Whitman–Broer–Kaup's type system [33]. In Section 4, we give a brief summary of our results.

2. Preliminaries

Before embarking into the details of an invariant subspace method to time fractional scalar and coupled nonlinear PDEs, we would like to recall some basic definitions, results and properties of the fractional calculus operators, used in the remaining part of the article.

Definition 2.1. The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of the function $f(t) \in L^1([a,b],\mathbb{R}_+)$, denoted by $I^{\alpha}_{a^+}$, is defined by [1,2]

$$I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau,\ t > a, \qquad (1)$$
$$I^{0}_{a^+} f(t) = f(t),$$

where $\Gamma(\lambda) = \int_0^{\infty} x^{\lambda-1}e^{-x}dx$ is the Euler Gamma-function. For $n \in \mathbb{N}$, we denote by $AC^n([a, b])$ the space of functions $f(t)$ which have continuous derivative up to order $(n-1)$ on $[a, b]$ such that $f^{(n-1)}(t) \in AC([a,b])$.

Definition 2.2. The Caputo fractional differential operator of order $\alpha > 0$ of the function $f(t) \in AC([a, t])$, denoted by $D^{\alpha}_{a^+}$, is defined by [1]

$$D^{\alpha}_{a^+} f(t) = I^{n-\alpha}_{a^+} D^n f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{(n-\alpha-1)} f^{(n)}(\tau)d\tau, & \text{if } n-1 < \alpha < n,\ n \in \mathbb{N} \\ f^{(n)}(t), & \text{if } \alpha = n \in \mathbb{N} \end{cases} \qquad (2)$$

for $t > a$.

If $\alpha = 0$, then $D^{\alpha}_{a^+} f(t) = f(t)$.

In this article, the Caputo definition of fractional differentiation is used. Note that the Caputo derivative of real order introduces a memory formalism as it is an integro-differential operator defined by the convolution of the ordinary derivative with a power law kernel.

For simplicity, we denote the operators $D^{\alpha}_{0+} f(t)$ and $I^{\alpha}_{0+} f(t)$ respectively as $D^{\alpha}f(t)$ and $I^{\alpha}f(t)$. The Caputo fractional derivative and Riemann–Liouville fractional integral satisfy the following properties(see[39]):

$$D^{\alpha}[f(t)+g(t)] = D^{\alpha}f(t) + D^{\alpha}g(t),$$
$$D^{\alpha}I^{\alpha}f(t) = f(t),\quad \alpha > 0,$$
$$I^{\alpha}D^{\alpha}f(t) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k,\quad \alpha > 0, t > 0$$
$$I^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}t^{\beta+\alpha},\ \alpha > 0,\ \beta > -1,\ t > 0,$$