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Research paper

On the local fractional derivative of everywhere non-differentiable continuous functions on intervals

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ABSTRACT

We first prove that for a continuous function $f(x)$ defined on an open interval, the Kolvankar-Gangal's (or equivalently Chen-Yan-Zhang's) local fractional derivative $f^{(\alpha)}(x)$ is not continuous, and then prove that it is impossible that the KG derivative $f^{(\alpha)}(x)$ exists everywhere on the interval and satisfies $f^{(\alpha)}(x) \neq 0$ in the same time. In addition, we give a criterion of the nonexistence of the local fractional derivative of everywhere non-differentiable continuous functions. Furthermore, we construct two simple nowhere differentiable continuous functions on (0, 1) and prove that they have no the local fractional derivatives everywhere.

1. Introduction

There are a large number of literatures on fractional calculus (see, for example, [1,2] and the references therein). For several famous fractional derivatives such as Riemann-Liouville, Caputo, Weyl, Riesz and others, their properties are different from the usual integer-order derivative. In particular, Leibniz rule $D^\alpha(fg) = D^\alpha(f)g + fD^\alpha(g)$ is not again satisfied where $0 < \alpha < 1$. Some authors try to give new definitions of the fractional derivatives to satisfy the Leibniz rule(see, for example, Refs. [3–8]). However, there are counterexamples to show that the Leibniz rule is not satisfied by the Jumarie's fractional derivative [9]. In addition, in passing, now I want to give some comments on two so-called conformable fractional derivatives [5,6]. One of these two definitions is given as follows [5]

$$T_\alpha(f)(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \tag{1}$$

and another is given by [6]

$$D^{(\alpha)}(f)(t) = \lim_{\epsilon\to 0} \frac{f(te^{\epsilon t^{-\alpha}}) - f(t)}{\epsilon}, \tag{2}$$

where $t > 0$ and $\alpha \in (0, 1)$. If expanding $e^{\epsilon t^{-\alpha}}$ at $\epsilon = 0$, the definition (2) will be transformed to the definition (1). It is obvious that these two derivatives satisfy the Leibniz rules and other main formulas in usual calculus. Since there exists the parameter α, these authors consider their definitions as new fractional derivatives namely conformable derivatives [5,6]. But, these so-called conformable derivatives are just the one-dimensional directional derivatives in usual calculus, only with

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different directions at the different points. On the other hand, for an arbitrary function $h(t)$, we can define a so-called h-derivative as

$$T_h(f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon h(t)) - f(t)}{\epsilon}. \tag{3}$$

If $h(t) = t^{\alpha-1}$, we get the T_α, and $h(t) = \frac{t}{\epsilon}(e^{\epsilon t^{-\alpha}} - 1)$ gives $D^{(\alpha)}$. Moreover, if f is differentiable, we have

$$T_h(f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon h(t)) - f(t)}{\epsilon} = f'(t)h(t). \tag{4}$$

This is just the usual derivative in the meaning of the Tarasov's theorem [10]. Therefore, the so-called conformable derivatives can not be considered as the fractional derivatives at all, and of course, the related studies are meaningless.

The Tarasov's theorem is an important result which shows that no violation of the Leibniz rule will be no fractional derivative for differentiable functions [10]. The result also mean that we need to consider everywhere non-differentiable functions and study whether or not they satisfy the Leibniz rule for some new definitions of the fractional derivatives. Furthermore, Tarasov [11,12] studied the Leibniz rules for some local fractional derivatives. Ortigueira and Machado [13] studied the Leibniz rule for the Riesz potential.

In the paper, our aim is to study the following Kolvankar-Gangal's local fractional derivative. Based on the Riemann-liouville definition, Kolvankar and Gangal defined a local fractional derivative [7,8],

$$D^\alpha f(x_0) = \lim_{x \to x_0} \frac{d^\alpha (f(x) - f(x_0))}{d(x-x_0)^\alpha}, \tag{5}$$

where the derivative $\frac{d^\alpha}{d(x-x_0)^\alpha}$ is the Riemann-Liouville derivative. Adda and Cresson [14,15] studied the properties of the local fractional derivative of non-differentiable functions. Chen et al [16] proved that the Kolvankar-Gangal's local fractional derivative can equivalently be defined by difference quotient

$$f^{(\alpha)}(x) = \Gamma(1+\alpha) \lim_{y \to x^\pm} \frac{\pm(f(y) - f(x))}{|y-x|^\alpha}. \tag{6}$$

Our starting point is just the Chen-Yan-Zhang's definition (6). It is obvious that this local fractional derivative satisfies the Leibniz rule and other rules such as $D^\alpha(f(g(x)) = f'(g)D^\alpha g(x)$. In addition, it is easy to see that for the smooth functions or evenly the γ-Hölderian ($\gamma > \alpha$) functions, the local fractional derivatives will be zero. Therefore, in order to get the nontrivial (or equivalently, nonzero) local fractional derivatives, we must consider the everywhere non-differentiable continuous functions on open intervals. However, a simple computation tells us that such functions perhaps do not exist at all. In fact, if a nonzero local fractional derivative exists at a point x, the usual derivative at the point will be infinity. This means that the graph of the function has a vertical tangent line at the point [17]. Intuitively, it is impossible that a continuous function has infinity derivative at every point on an open interval. In the present paper, I will give a strict proof to this conclusion. This result tells us that the nontrivial local fractional derivative of a nowhere differentiable function does not exist everywhere on an open interval. In other words, the set of points at which the local fractional derivatives exist is very small in a sense. For example, in [17], Li et al constructed a function on a fractal set such that its local fractional derivative is nontrivial. Thus a natural problem is whether or not a nowhere differential function will have a set of local differentiable points, which still includes enough many points, although the set is small. However, we give a criterion of the nonexistence of the local fractional derivative of non-differentiable functions, by which we prove that some nowhere differentiable functions have not the local fractional derivatives everywhere. As examples, I construct two simple nowhere differentiable continuous functions on (0, 1) and prove they have no the local fractional derivatives everywhere. Our results mean that the local fractional derivative is not a suitable tool to study the continuous functions on intervals.

This paper is organized as follows. In Section 2, we give three theorems and prove them in details. In Section 3, we use a simple method to construct two nowhere differentiable continuous functions on (0, 1), and prove that they are not the local fractional differentiable everywhere. In the last section, we give some discussions.

2. The main theorems

Theorem 1. *For a continuous and nowhere differentiable function $f(x)$ defined on an open interval I, if the local fractional derivative $f^{(\alpha)}(x)$ exists on I, and $f^{(\alpha)}(x) \neq 0$ for every $x \in I$ where $0 < \alpha < 1$, then $f^{(\alpha)}(x)$ is not continuous.*

Proof. We use contradiction to prove it. Without loss of generality, we assume $f^{(\alpha)}(x)$ is continuous at the point x_0 and $f^{(\alpha)}(x_0) > 0$. By the definition (6) of the local fractional derivative, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = +\infty. \tag{7}$$

By the continuity of $f^{(\alpha)}(x)$ at the point x_0, there exists a neighborhood $(a, b) \subset I$ of x_0 such that $f^{(\alpha)}(x) > 0$ for every $x \in (a, b)$. This means that the usual derivative $f'(x) = +\infty$ on the interval (a, b). We prove in the following it is impossible. In fact,