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Research paper

# Symmetry reduction related with nonlocal symmetry for Gardner equation

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ABSTRACT

Based on the truncated Painlevé method or the Möbious (conformal) invariant form, the nonlocal symmetry for the (1+1)–dimensional Gardner equation is derived. The nonlocal symmetry can be localized to the Lie point symmetry by introducing one new dependent variable. Thanks to the localization procedure, the finite symmetry transformations are obtained by solving the initial value problem of the prolonged systems. Furthermore, by using the symmetry reduction method to the enlarged systems, many explicit interaction solutions among different types of solutions such as solitary waves, rational solutions, Painlevé II solutions are given. Especially, some special concrete soliton-cnoidal interaction solutions are analyzed both in analytical and graphical ways.



## 1. Introduction

In nonlinear science, the investigation of exact solutions for nonlinear evolution equations is one of the most important problems. Many effective methods have been proposed, such as the inverse scattering transformation [1], the Hirota's bilinear method [2], symmetry reductions [3], the Darboux transformation [4], the Painlevé analysis method [5], the Bäklund transformation (BT) [6], the separated variable method [7], etc [8]. Among these traditional methods, it is difficult to obtain the interaction solutions among different nonlinear excitations [9]. However, the solitary waves must interact with other waves in the real physics world. How to find these interaction solutions is important topic in nonlinear science. Recently the localization procedure related with the nonlocal symmetry to find these types of interaction solutions has been proposed [10–12].

The aim of this paper is to explore the nonlocal symmetry of the Gardner equation and its applications. The finite symmetry transformations related with the nonlocal symmetry are obtained in the enlarged systems. The interaction solutions among solitons and other complicated waves including the Painlevé waves and periodic cnoidal waves of the Gardner equation are derived by the symmetry reduction method. Those interaction solutions are hard to obtain with other traditional methods.

The structure of this paper is as follows. In Section 2, the nonlocal symmetry for the Gardner equation is obtained with the truncated Painlevé method or the Möbious (conformal) invariant form. To obtain the finite symmetry transformations related by the nonlocal symmetry, the nonlocal symmetry for the original Gardner equation is localized by prolongation the Gardner equation. The finite symmetry transformations are thus obtained by solving the initial value problem of the Lie's first principle. In Section 3, the symmetry reductions for the extended systems are considered according to the Lie point

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symmetry theory. The corresponding interaction solutions are given with the similar reductions. The last section is a short summary and discussion.

## 2. Nonlocal symmetry and its localization for Gardner equation

The (1+1)–dimensional Gardner equation reads

$$u_t + u_{xxx} - 6\beta^2 u^2 u_x + \alpha u u_x = 0, \tag{1}$$

where $\alpha$ and $\beta$ arbitrary constants [13]. (1) can be applied in dusty plasma, ocean and atmosphere, fluid mechanics and solid state physics [14–16]. Various methods for studying integrability properties and exact solutions of the Gardner equation have been reported. [17–23]. In this section, we shall study the nonlocal symmetry and its localization process for the Gardner Eq. (1).

According to the truncated Painlevé analysis of the Gardner equation, the Laurent series expansion of $u$ is [5]

$$u = \frac{u_0}{\phi} + u_1, \tag{2}$$

where the function $\phi$, $u_0$ and $u_1$ are analytic functions in a neighborhood of the noncharacteristic singular manifold. By substituting of expansion (2) into (1) and vanishing the coefficient of $\phi^{-4}$ independently, we obtain

$$u_0 = \pm\frac{\phi_x}{\beta}. \tag{3}$$

We proceed further and collect the coefficient of $\phi^{-3}$ to get

$$u_1 = -\frac{1}{2\beta}\frac{\phi_{xx}}{\phi_x} + \frac{\alpha}{12\beta^2}. \tag{4}$$

Substituting the expressions (2), (3) and (4) into (1), the field $\phi$ satisfies the following Schwarzian Gardner form

$$\frac{\phi_t}{\phi_x} + \{\phi; x\} + \frac{\alpha^2}{24\beta^2} = 0, \tag{5}$$

where $\{\phi; x\} = \frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2$ is the Schwarzian derivative.

Based on the definition of residual symmetry (RS) [12], the nonlocal symmetry of the Gardner Eq. (1) can be read out from the truncated Painlevé analysis (2)

$$\sigma^u = \pm\frac{\phi_x}{\beta}. \tag{6}$$

The nonlocal symmetry (6) can be also obtained by using (5) and (4) [24,25]. The Schwarzian form (5) is invariant under the Möbious transformation [5]

$$\phi \to \frac{a\phi + b}{c\phi + d}, \qquad ac \neq bd. \tag{7}$$

It means (5) possesses the symmetry

$$\sigma^\phi = a\phi + a\phi^2, \tag{8}$$

where the constants are $d = 1,\ b = 0,\ c = -\epsilon$ in (7). The nonlocal symmetry (6) will be obtained with substituting the Möbious transformation symmetry (8) into the symmetry equation of (4).

According to the Lie's first principle, the initial value problem related with the nonlocal symmetry (6) will be expressed

$$\frac{d\bar{u}}{d\epsilon} = \pm\frac{\phi_x}{\beta}, \quad \bar{u}|_{\epsilon=0} = u. \tag{9}$$

It is difficult to solve the initial value problem of the Lie's first principle (9) due to the intrusion of the function $\phi$ and its differentiation [12]. To solve the initial value problem (9), one can localize the nonlocal symmetry to the local Lie point symmetry for the prolonged systems [12]. The space derivative of field $\phi$ can be eliminated by introducing the following potential field

$$\phi_x = g. \tag{10}$$

It is easily verified that the solution of the symmetry equation for the prolonged systems (1), (4) and (10) gives

$$\sigma^u = \frac{g}{\beta}, \quad \sigma^\phi = -\phi^2, \quad \sigma^g = -2\phi g. \tag{11}$$