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Research paper

The fifth-order partial differential equation for the description of the $\alpha + \beta$ Fermi–Pasta–Ulam model

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A B S T R A C T

We study a new nonlinear partial differential equation of the fifth order for the description of perturbations in the Fermi–Pasta–Ulam mass chain. This fifth-order equation is an expansion of the Gardner equation for the description of the Fermi–Pasta–Ulam model. We use the potential of interaction between neighbouring masses with both quadratic and cubic terms. The equation is derived using the continuous limit. Unlike the previous works, we take into account higher order terms in the Taylor series expansions. We investigate the equation using the Painlevé approach. We show that the equation does not pass the Painlevé test and can not be integrated by the inverse scattering transform. We use the logistic function method and the Laurent expansion method to find travelling wave solutions of the fifth-order equation. We use the pseudospectral method for the numerical simulation of wave processes, described by the equation.

1. Introduction

The Fermi–Pasta–Ulam (FPU) model was first studied in work [1]. It describes perturbations in the chain of nonlinear coupling among masses. It is shown [1] that energy remains in a very few modes and the long-time dynamics of the system is recurrent. This fact was called the FPU paradox. A set of approaches was used to explain it (see for example [2–7]). The main ideas of these approaches and their results can be found in work [8].

Usually α or β FPU model is considered in literature, but in paper [9] the $\alpha + \beta$ FPU model is investigated. In this paper, we also use the $\alpha + \beta$ model but we use the continuous limit approximation for the investigation.

This approach was used for the first time by N. Zabusky and M. Kruskal in work [10], where the authors derived the Korteweg–de Vries (KdV) equation for the description of the α FPU model. This equation is integrable by the inverse scattering transform and has been widely studied by now ([11,12]). Zabusky and Kruskal explained the FPU paradox (recurrence of initial conditions) using the numerical simulation of the KdV equation. Another result of work [10] was an introduction of soliton. In article [13] the author took into account high order (in comparison with [10]) terms in the Taylor series expansions for the continuous limit approximation of the α FPU model. The generalized KdV equation was obtained for a more accurate description of wave processes in the FPU model. Some exact solutions of the derived equation were found . It is shown [13] that if one takes into account more terms in the Taylor series and acts in analogy to paper [10] then he can not derive an integrable equation for the description of wave processes in the FPU mass chain. Recurrence of initial statement

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is also unrepresentative for wave processes, described by the fifth-order equation with arbitrary initial conditions. In paper [14] wave processes for the generalized KdV equation are simulated numerically.

This brings up the question, whether the situation is similar for the β and the $\alpha+\beta$ FPU models? It is known [11] that one can obtain the Gardner equation for the description of the $\alpha+\beta$ FPU model using the continuous limit. The Gardner equation is integrable by the inverse scatterring transform and can be reduced to the modified KdV equation [15]. If we take more terms in the Taylor series expansion, we can derive the fifth-order partial differential equation for the description of the $\alpha+\beta$ FPU model. To the best of our knowledge, this equation has not been obtained and investigated before. This is the main aim of the current manuscript.

The rest of the work is organized as follows. In Section 2 we derive the fifth-order partial differential equation for the description of the $\alpha+\beta$ FPU model. In Section 3 we use Painlevé test to analyze the derived equation. The logistic function method is used in Section 4 to find exact solutions of the derived equation. In Section 5 we obtain an elliptic solution for this equation. In Section 6 we present the results for the numerical simulation of wave processes, described by the derived equation. In Section 7 we briefly discuss our results.

2. The fifth-order partial differential equation for the description of the $\alpha+\beta$ Fermi–Pasta–Ulam model

Let us consider the $\alpha+\beta$ Fermi–Pasta–Ulam model. It appears as the following system of equations:

$$\begin{aligned} m\frac{d^2y_i}{d\tau^2} &= \gamma(y_{i+1}-y_i)+\alpha(y_{i+1}-y_i)^2-\beta(y_{i+1}-y_i)^3-\gamma(y_i-y_{i-1})-\alpha(y_i-y_{i-1})^2+\beta(y_i-y_{i-1})^3 \\ &= (y_{i+1}-2y_i+y_{i-1})(\gamma+\alpha(y_{i+1}-y_{i-1})-\beta(y_{i+1}^2+y_i^2+y_{i-1}^2-y_iy_{i+1}-y_iy_{i-1}-y_{i+1}y_{i-1})), \end{aligned} \tag{1}$$

where y_i is a displacement of the particle number i from its equilibrium position, i runs through values from 1 to N, where N is the number of particles, m is the mass of a single particle, α,β,γ are some positive constants that characterize the potential of interaction and τ is the time. Let us find the continuous limit approximation of the model. In order to do this we use the following Taylor series expansions:

$$y_{i\pm1}=y\pm hy_\xi+\frac{h^2}{2}y_{\xi\xi}\pm\frac{h^3}{6}y_{\xi\xi\xi}+\frac{h^4}{24}y_{\xi\xi\xi\xi}\pm\frac{h^5}{120}y_{\xi\xi\xi\xi\xi}+\frac{h^6}{720}y_{\xi\xi\xi\xi\xi\xi}+\dots,$$

where $y=y(\xi,\tau)$ is a displacement of the infinitely small part of the chord with coordinate ξ in time moment τ, h is a small parameter. Substituting expansion (2) into the system of Eqs. (1) we derive a partial differential equation for the description of the system dynamics in the continuous limit approximation. Let us take into account only terms up to h^6.

$$\begin{aligned} m\frac{d^2y}{d\tau^2} &= h^2\gamma y_{\xi\xi}+2\alpha h^3y_\xi y_{\xi\xi}-3\beta h^4y_\xi^2y_{\xi\xi}+\frac{\gamma h^4}{12}y_{\xi\xi\xi\xi}+\frac{\alpha h^5}{3}y_{\xi\xi}y_{\xi\xi\xi} \\ &+\frac{\alpha h^5}{6}y_\xi y_{\xi\xi\xi\xi}-\beta h^6y_\xi y_{\xi\xi}y_{\xi\xi\xi}-\frac{\beta h^6}{4}y_{\xi\xi}^3-\frac{\beta h^6}{4}y_\xi^2y_{\xi\xi\xi\xi}+\frac{\gamma h^6}{360}y_{\xi\xi\xi\xi\xi\xi}. \end{aligned} \tag{3}$$

Using a new parameter:

$$c^2=\frac{\gamma h^2}{m}, \tag{4}$$

and scaling transformations:

$$y=\frac{\gamma\sqrt{\gamma}}{2c\alpha\sqrt{m}}y',\quad T=\frac{c}{2}\tau, \tag{5}$$

we reduce Eq. (3) to the form:

$$\begin{aligned} \frac{1}{4}\frac{d^2y}{dT^2} &= y_{\xi\xi}+y_\xi y_{\xi\xi}-\mu y_\xi^2y_{\xi\xi}+\delta y_{\xi\xi\xi\xi}+2\delta y_{\xi\xi}y_{\xi\xi\xi}+\delta y_\xi y_{\xi\xi\xi\xi} \\ &-4\mu\delta y_\xi y_{\xi\xi}y_{\xi\xi\xi}-\mu\delta y_{\xi\xi}^3-\mu\delta y_\xi^2y_{\xi\xi\xi\xi}+\frac{2}{5}\delta^2y_{\xi\xi\xi\xi\xi\xi}, \end{aligned} \tag{6}$$

where $\mu=\frac{3\beta\gamma}{4\alpha^2}$, $\delta=\frac{mc^2}{12\gamma}$ and primes are omitted. According to [11], the solution of this equation can be found as two oppositely directed running waves. Following [10], we only study the wave propagating to the right. For this purpose we search for the solution of Eq. (6) in the following form:

$$y(\xi,T)=f(x,t)+\varepsilon y_1(\xi,T),\quad x=\xi-2T,\quad t=\varepsilon T. \tag{7}$$

Here $f(x,t)$ corresponds to the wave profile on long distance. We omit members that have high order in epsilon. Then from Eq. (6), we derive an equation for $f(x,t)$:

$$f_{xt}+f_xf_{xx}-\mu f_x^2f_{xx}+\delta f_{xxxx}+2\delta f_{xx}f_{xxx}+\delta f_xf_{xxxx}-4\mu\delta f_xf_{xx}f_{xxx}-\mu\delta f_{xx}^3-\mu\delta f_x^2f_{xxxx}+\frac{2}{5}\delta^2f_{xxxxxx}=0. \tag{8}$$

Let us use a new variable:

$$u(x,t)=f_x(x,t), \tag{9}$$