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Research paper

# Symmetry classification of time-fractional diffusion equation

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ABSTRACT

In this article, a new approach is proposed to construct the symmetry groups for a class of fractional differential equations which are expressed in the modified Riemann-Liouville fractional derivative. We perform a complete group classification of a nonlinear fractional diffusion equation which arises in fractals, acoustics, control theory, signal processing and many other applications. Introducing the suitable transformations, the fractional derivatives are converted to integer order derivatives and in consequence the nonlinear fractional diffusion equation transforms to a partial differential equation (PDE). Then the Lie symmetries are computed for resulting PDE and using inverse transformations, we derive the symmetries for fractional diffusion equation. All cases are discussed in detail and results for symmetry properties are compared for different values of $\alpha$. This study provides a new way of computing symmetries for a class of fractional differential equations.



## 1. Introduction

The fractional differential equations arise in the study of fractals, acoustics, control theory, signal processing [1,2] and many other important studies such as in physical, chemical and biological processes [3–5]. One may encounter various processes in science and engineering where both spatial and temporal variations occur which result in a phenomena known as diffusion. Such processes include heat conduction in materials, transient flow in porous media, dispersion of chemicals and pollutants in its surrounding by gradual decrease in their concentration, transport of substances through cell membrane in cell biology, sedimentation and consolidation of geomaterials etc. In fact diffusion is a process in which molecules move around until they are evenly spread out in the area. Diffusion is often described by the power law $r^2(t) = Dt^{\alpha}$, where $D$ is the diffusion coefficient, $t$ is the elapsed time and $r^2(t)$ is the mean squared displacement. For $\alpha > 1$, the phenomenon is referred as a super diffusion and for $\alpha = 1$, it is called a normal diffusion whereas $\alpha < 1$ describes the subdiffusion. When $\alpha < 1$ or $\alpha > 1$, the diffusion process is anomalous due to nonlinear relationship with time. Often this irregularity is because of either nonhomogeneous medium or active cellular transport. There are number of frameworks to describe anomalous diffusion that are currently in fashion in statistical physics and biology which includes continuous time random walk (CRTW) and fractional Brownian motion. We consider the time-fractional diffusion equation [6]

$$ {}^{R}D_t^{\alpha} u = (k(u)u_x)_x, \quad 0 < \alpha \le 2, \tag{1}$$

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where ${}^{R}D_{t}^{\alpha}$ is the Riemann-Liouville fractional differential operator of order $\alpha$ defined as

$$ {}^{R}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x f(\xi)(x-\xi)^{n-\alpha-1} d\xi, \qquad n-1 < \alpha \le n, \quad n \in \mathbb{N}. \tag{2} $$

In Eq. (2), $f \in \mathcal{C}_{\alpha}^{n},\ x > 0,\ \ \alpha \in \mathcal{R}$ and there exists a real number $\beta > \alpha$ such that $f(x) = x^{\beta} g(x)$, where $g(x) \in \mathcal{C}[0, \infty)$. The fractional diffusion Eq. (1) investigates the mechanism of anomalous diffusion that arise in transport processes through complex and/or disordered systems including fractal media.

The definition of a fractional differential operator is not unique. The fractional differential equations can be expressed in terms of different differential operators defined by Riemann-Liouville [7], Caputo [8,9], Weyl [10] and many others (see e.g. [1,2,10] and references therein). Guy Jumaire [10] proposed some modifications in Riemann-Liouville fractional derivative and derived the fractional Taylor series of non differentiable functions. The new modified fractional derivative has some features similar with the classical derivative. The modified fractional derivatives are defined by

$$ {}^{J}D_{x}^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^x (f(\xi) - f(0))(x-\xi)^{-\alpha-1} d\xi, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (f(\xi) - f(0))(x-\xi)^{-\alpha} d\xi, & 0 < \alpha \le 1, \\ (f^{(n-1)}(x))^{(\alpha-n+1)}, & n-1 < \alpha \le n, \quad n \ge 2. \end{cases} \tag{3} $$

The modified Riemann-Liouville fractional derivative bears some interesting properties:

$$ {}^{J}D_{x}^{\alpha} x^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha}, \qquad \mu > 0, \tag{4} $$

$$ {}^{J}D_{x}^{\alpha} (f(x)g(x)) = g(x)\, {}^{J}D_{x}^{\alpha} f(x) + f(x)\, {}^{J}D_{x}^{\alpha} g(x), \tag{5} $$

$$ {}^{J}D_{x}^{\alpha} f[g(x)] = \frac{df[g(x)]}{dg(x)}\, {}^{J}D_{x}^{\alpha} g(x). \tag{6} $$

In Eq. (1), the fractional order $0 < \alpha < 1$ describes the sub-diffusion. Eq. (1) has been considered by various researchers. Numerous methods have been developed and used to find approximate and analytical solutions of these equations e.g. Laplace transform method, Green's function method [1], variational iteration method [11], Adomian decomposition method [12], Finite Sine transform method [13], homotopy perturbation method [14] and symmetries method [15]. The fundamental solution was obtained by Mainardi [16]. Wyss [17] obtained solutions to the Cauchy problem in terms of H-functions using the Mellin transform. Schneider and Wyss [18] converted the diffusion-wave equation with appropriate initial conditions into the integro-differential equation and found the corresponding Green functions in terms of Fox functions. Gazizov et. al. [6] considered the fractional diffusion equation with nonlinear diffusion coefficient and provided the complete Lie group classification ($0 < \alpha \le 2$) using Riemann- Liouville and Caputo fractional derivatives. The over-determined system of linear FDEs was obtained in the classification which was solved for infinitesimals. The invariance of a partial differential equation of fractional order under the Lie group of scaling transformations was studied by Buckwar and Luchko [19]. The point transformations of variables in fractional integrals and derivatives of different types were discussed by Gazizov et. al. [20]. They used the prolongation formulae for finding nonlocal symmetries of ordinary fractional differential equations. The algorithm for the computation of Lie point symmetries for fractional order differential equations using the method described by Buckwar and Luchko and Gazizov, Kasatkin and Lukashchuk is developed by Jefferson et. al. [21]. The method was generalized to calculate symmetries for FDEs with $n$ independent variables and also for systems of partial FDEs. The systematic way to compute solutions for the linear time-fractional diffusion wave equation are presented in [22]. The integral transform technique is utilized which discusses the properties of the Mittag-Leffler, Wright and Mainardi functions that appear in the solution.

In this article, we provide the alternative way of computing symmetries which is more convenient and easier and provide more general results. These symmetries can be utilized to compute the invariant solutions for fractional differential equations. We first transform the Riemann-Liouville differential operator to the modified Riemann-Liouville fractional operator and then convert this operator in classical differential operator using suitable transformations. As a result an over-determined system of linear PDEs can be obtained after using invariance condition whose solution process is a well established procedure in the literature. Then solving these determining equations for unknown coefficients $\xi^1$, $\xi^2$, $\eta$ and using inverse transformation we obtain the infinitesimal generators for fractional diffusion equation. This method is successfully applied to the fractional diffusion equation which is considered in Riemann-Liouville sense when $0 < \alpha \le 2$.

This article is organized in the following manner. In Section 2, we provide a complete classification of fractional diffusion equation to construct symmetries. All cases are discussed in detail. We compare our result with ones obtained by Gazizov et. al. [6]. The physical interpretation of symmetry properties of fractional diffusion equation is discussed in Section 3. Section 4 is devoted to the concluding remarks.