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Research paper

Comprehensive analysis of the symmetries and conservation laws of the geodesic equations for a particular string inspired FRLW solution

Fatemeh Ahangari*

Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, P.O.Box 1993893973, Iran

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ABSTRACT

Scalar-field cosmology can be regarded as one of the significant fields of research in recent years. This paper is dedicated to a thorough investigation of the symmetries and conservation laws of the geodesic equations associated to a specific exact cosmological solution of a scalar-field potential which was originally motivated by six-dimensional Einstein-Maxwell theory. The mentioned string inspired Friedmann-Robertson-Lamaître-Walker (FRLW) solution is one of the noteworthy solutions of Einstein field equations. For this purpose, first of all the Christoffel symbols and the corresponding system of geodesic equations are computed and then the associated Lie symmetries are totally analyzed. Moreover, the algebraic structure of the Lie algebra of local symmetries is briefly investigated and a complete classification of the symmetry subalgebras is presented. Besides by applying the resulted symmetry operators the invariant solutions of the system of geodesic equations are discussed. In addition, the Noether symmetries and the Killing vector fields of the geodesic Lagrangian are determined and the corresponding optimal system of one-dimensional subalgebras is constructed. Mainly, an entire set of local conservation laws is computed for our analyzed scalar-field cosmological solution. For this purpose, two distinct procedures are applied: the celebrated Noether's theorem and the direct method which is fundamentally based on a systematic application of Euler differential operators which annihilate any divergence expression identically.

1. Introduction

Scalar-field cosmology has been discussed extensively in recent years and indeed a lot of attention has been paid in the literature regarding this topic. In addition, various researches have illustrated that certain gravitational field equations in scalar field cosmology can be reformulated in terms of nonlinear Schrodinger-type (NLS) equations or generalized Ermakov-Milne-Pinney (EMP) equations (e.g. [11,12,22,27]). Taking into account the fact that these equations arise considerably in distinct physical subjects, it is reasonable to observe them appearing in gravitational models as well. It is worth mentioning that some recent studies have been dedicated to recast computational problems in the equivalent NLS or EMP language. Moreover, EMP/NLS can be significantly applied in order to determine mappings between gravitational and non-gravitational

* Fax +9821 88035187.
E-mail address: f.ahangari@alzahra.ac.ir, fa.ahangari@gmail.com

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systems. For instance, in [14–16] such reformulations and mappings of Einstein's gravitational field equations to nonzero curvature and higher dimensional models are comprehensively investigated.

Einstein's gravitational field equations (EFE) can be regarded as the fundamental equations of general relativity in $d+1$ spacetime dimensions and are formulated as the following tensorial equations:

$$G_{ij} = -\kappa T_{ij} + \Lambda g_{ij}, \qquad i, j \in \{0, \dots, d\}. \tag{1.1}$$

where κ is a generalization of $8\pi G$, for G Newton's constant to $d+1$ spacetime dimensions and Λ is the cosmological constant. Furthermore, the Einstein tensor is defined by: $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$, where R_{ij} is known as the Ricci tensor, R is the scalar curvature and g_{ij} is the corresponding metric. The Ricci tensor is defined by:

$$R_{ij} = \sum_{k=0}^{d}\left(\Gamma^k_{kj,i} - \Gamma^k_{ij,k}\right) + \sum_{m=0}^{d}\sum_{n=0}^{d}\Gamma^n_{im}\Gamma^m_{nj} - \sum_{m=0}^{d}\sum_{n=0}^{d}\Gamma^m_{ij}\Gamma^n_{nm}, \qquad i, j = \{0, 1, \cdots, d\}. \tag{1.2}$$

where Γ^k_{ij} are the Christoffel symbols of the second kind $\Gamma^k_{ij} = \frac{1}{2}\sum_{s=0}^{d} g^{sk}\left(g_{si,j} - g_{ij,s} + g_{js,i}\right)$. Furthermore, the energy-momentum tensor is considered as the sum of two terms: $T_{ij} = T^{(1)}_{ij} + T^{(2)}_{ij}$. The first term is regarded as the energy-momentum tensor for a minimally coupled scalar field $\mathbf{\Phi}$ with potential $\mathbf{V}$, consequently we have:

$$T^{(1)}_{ij} = \partial_i\mathbf{\Phi}\ \partial_j\mathbf{\Phi} - g_{ij}\left[\frac{1}{2}\sum_{k=0}^{d}\partial^k\mathbf{\Phi}\ \ \partial_k\mathbf{\Phi} + \mathbf{V}\circ\mathbf{\Phi}\right], \tag{1.3}$$

where $\partial^k\mathbf{\Phi} = \sum_{l=0}^{d} g^{kl}\partial_l\mathbf{\Phi}$. By considering $\mathbf{\Phi}(t)$ to be dependent only on time $x_0 = t$ the Eq. (1.3) reduces to the following relation:

$$T^{(1)}_{ij} = \delta_{0i}\delta_{0j}\dot{\mathbf{\Phi}}^{\mathbf{2}} - g_{ij}\left[\frac{1}{2}g^{00}\dot{\mathbf{\Phi}}^2 + \mathbf{V}\circ\mathbf{\Phi}\right], \qquad \delta_{0i} = \begin{cases}1 & \text{if} \quad i = 0,\\ 0 & \text{if} \quad i \neq 0.\end{cases} \tag{1.4}$$

Moreover, for density and pressure functions $\rho(t)$ and $p(t)$ the second term of the energy-momentum tensor is defined by:

$$T^{(2)} = \begin{pmatrix} -\rho(t)g_{00} & & & & \\ & p(t)g_{11} & & & \\ & & p(t)g_{22} & & \\ & & & \ddots & \\ & & & & p(t)g_{dd} \end{pmatrix} \tag{1.5}$$

Meanwhile, the off-diagonal entries are zero. It is noticeable that in the particular case when the metric is diagonal and $g_{00} = -1$, according to (1.4) it can be demonstrated that:

$$T^{(1)}_{00} = \frac{1}{2}\dot{\mathbf{\Phi}}^2 + \mathbf{V}\circ\mathbf{\Phi}, \qquad T^{(1)}_{ii} = g_{ii}\left(\frac{1}{2}\dot{\mathbf{\Phi}}^2 - \mathbf{V}\circ\mathbf{\Phi}\right), \quad 1 \le i \le d.$$

In other words, $T^{(1)}_{ij}$ reduces to the energy-momentum tensor for a perfect fluid with density and pressure respectively given by:

$$\rho_{\mathbf{\Phi}}(t) = \frac{1}{2}\dot{\mathbf{\Phi}}^2 + \mathbf{V}\circ\mathbf{\Phi}, \quad \text{and} \quad p_{\mathbf{\Phi}}(t) = \frac{1}{2}\dot{\mathbf{\Phi}}^2 - \mathbf{V}\circ\mathbf{\Phi}$$

in terms of the scalar field and potential. It is worth mentioning that considering the fact that the field equations are a coupled nonlinear system of partial differential equations whose solutions can not be generally known, it is essential to impose symmetries or any other physical assumptions on the metric and the energy-momentum tensor T_{ij}. Application of a real scalar field $\mathbf{\Phi}(t)$ and potential $\mathbf{V}(\mathbf{\Phi}(t))$ can be regarded as one of the fruitful ways in which nonzero energy density is imparted on a spacetime manifold.

In a first approximation our universe is modeled by applying the homogeneous and isotropic Friedmann-Robertson-Lamaître-Walker (FRLW) cosmological model which is defined by the following metric in $d+1$ dimensions:

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2_{d-1}\right), \quad d\Omega^2_{d-1} = d\theta_1^2 + \sin^2\theta_1\ d\theta_2^2 + \cdots + \sin^2\theta_1\cdots\sin^2\theta_{d-2}\ d\theta^2_{d-1}. \tag{1.6}$$

where $a(t)$ is denoted as the scale factor, $k \in \{-1, 0, 1\}$ is the curvature parameter.

Now the energy density and pressure in Eq. (1.5) are respectively considered as the following [15]:

$$\rho(t) = \sum_{i=1}^{M}\frac{D_i(t)}{a(t)^{n_i}} + \rho'(t), \qquad p(t) = \sum_{i=1}^{M}\frac{(n_i - d)D_i(t)}{da(t)^{n_i}} + p'(t), \quad n_i \in \mathrm{R}, \quad 1 \le i \le M. \tag{1.7}$$