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Research paper

# A new fractional wavelet transform

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A R T I C L E I N F O

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A B S T R A C T

The fractional Fourier transform (FRFT) is a potent tool to analyze the time-varying signal. However, it fails in locating the fractional Fourier domain (FRFD)-frequency contents which is required in some applications. A novel fractional wavelet transform (FRWT) is proposed to solve this problem. It displays the time and FRFD-frequency information jointly in the time-FRFD-frequency plane. The definition, basic properties, inverse transform and reproducing kernel of the proposed FRWT are considered. It has been shown that an FRWT with proper order corresponds to the classical wavelet transform (WT). The multiresolution analysis (MRA) associated with the developed FRWT, together with the construction of the orthogonal fractional wavelets are also presented. Three applications are discussed: the analysis of signal with time-varying frequency content, the FRFD spectrum estimation of signals that involving noise, and the construction of fractional Harr wavelet. Simulations verify the validity of the proposed FRWT.



## 1. Introduction

Fourier transform (FT) is one of the most valuable and frequently used tools in signal processing and analysis. For FT, a signal can be represented either in the time or in the frequency domain, and it can be viewed as the time-frequency representation of a signal. However, the Fourier coefficients define the average spectral content over the entire duration of the signal and it does not give any information about the occurrence of the frequency component at a particular time and is not applicable for non-stationary signals. A straightforward approach to overcome this problem is to perform the FT on a block by block basis rather than to process the entire signal at once, which is named the short-time Fourier transform (STFT) [1]. Although STFT has rectified almost all the limitations of FT, but still in some cases STFT is also not applicable as in the case of real signals having low frequencies of long duration and high frequencies of short duration. Such signals could be better described by a transform which has a high time resolution for short-lived high-frequency phenomena, and has high frequency resolution for long-lasting low-frequency phenomena. In these types of situations, wavelet transform (WT) can provide a better description of the signal instead of the STFT. Since wavelets have special ability to analyze signal in both time and frequency domain simultaneously, and can easily detect the local properties of a signal, WT is widely used to analyze transient and non-stationary signals.

Recently, researchers have come up with the new transform namely fractional Fourier transform (FRFT). FRFT is the generalization of FT since it can be viewed as the rotation through an angle $\alpha$ of FT. Like FT corresponds to a rotation in the time-frequency plane over an angle $\alpha = \pi/2$, the FRFT corresponds to a rotation over an arbitrary angle $\alpha = p \times \pi/2$ with $p \in \mathbb{R}$. That is to say, FRFT is the representation of a signal in the fractional Fourier domain (FRFD) [2]. Although the

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FRFT has a number of attractive properties, the fractional Fourier representation of a signal only provides overall FRFD-frequency content with no indication about the occurrence of the FRFD spectral component at a particular time. Since the FRFT uses a global kernel like FT, it fails in locating the FRFD spectral contents which is required in some applications. So the representation combining the time and FRFD-frequency information should be developed which is termed the time-FRFD-frequency representation (TFFR). The short-time FRFT (STFRFT) is one of such approaches. The idea is segmenting the signal by using a time-localized window and performing FRFT spectral analysis for each segment [3]. Since the FRFT is computed for each windowed segment of the signal, the STFRFT can provide the time and FRFD spectral information jointly in the time-FRFD plane. Another kind of TFFR method is the fractional WT (FRWT). The concept of FRWT was initially proposed in [4], where FRFT is firstly used to derive the fractional spectrum of a signal and WT is then performed on the obtained fractional spectrum. Since the fractional spectrum derived by the FRFT only represents the FRFD-frequency over the entire duration of the signal, the FRWT defined in [4] actually fails in obtaining the information of the local property of the signal. In [5], a fractional wave packet transform was developed and the basic idea is to introduce the wavelet basis function to FRFT. More recently, a new FRWT was proposed in [6] based on the concept of fractional convolution. The multiresolution analysis (MRA) associated with this kind of FRWT was then given in [7] by the same authors. Since this kind of FRWT analyze the signal in time-frequency-FRFD domain, its physical meaning requires deeper interpretation. Another kind of FRWT which was developed in [8] solves the issue in [6] since the analysis only involves time-FRFD domain. However, the MRA associated with this kind of FRWT is not addressed. In [9], WT and FRWT are, respectively, used for the simultaneous spectral analysis of a binary mixture system.

The purpose of this paper is to define a new type of FRWT that has more elegant mathematical properties and is more general than the transforms defined in [6] and [8]. The new FRWT displays the time and FRFD-frequency information jointly in the time-FRFD-frequency plane. The remainder of the paper is organized as follows. In Section 2, preliminaries about FRFT and WT are given. In Section 3, the theoretical framework of FRWT is established, including its definition, properties, inverse transform and reproducing kernel equation. In Section 4, the MRA associated with the developed FRWT, together with the construction of the orthogonal fractional wavelets are described. Three applications are discussed in Section 5, including the process of non-stationary signal and the construction of fractional Harr wavelets. The last section concludes this paper and presents its future directions.

## 2. Preliminaries

### 2.1. Fractional fourier transform

Mathematically, the $\alpha$-order FRFT of a signal $x(t) \in L^2(\mathbb{R})$ is defined as

$$X_\alpha(u) = F^\alpha[x(t)] = \int_{-\infty}^{+\infty} x(t)K_\alpha(t,u)dt \tag{1}$$

where the transform kernel is given by

$$K_\alpha(t,u) = \begin{cases} A_\alpha e^{\frac{j}{2}(t^2+u^2)\cot\alpha - jtu\csc\alpha}, & \alpha \neq k\pi \\ \delta(t-u), & \alpha = 2k\pi \\ \delta(t+u), & \alpha = (2k+1)\pi \end{cases} \tag{2}$$

and $A_\alpha$ is given by

$$A_\alpha = \sqrt{\frac{1 - j\cot\alpha}{2\pi}} \tag{3}$$

The inverse FRFT is

$$x(t) = \int_{-\infty}^{+\infty} X_\alpha(u)K_\alpha^*(t,u)du \tag{4}$$

The definition implies that the FRFT is the decomposition into the chirp bases $\{K_\alpha^*(t,u)\}$. So a proper order FRFT of the chirp signal is an impulse. The argument $u$ represents a new physical quantity extended from the frequency concept and is termed the FRFD-frequency, so the FRFT can also be interpreted as the FRFD-spectrum [10]. The Parseval identity of FRFT, which will be useful in this paper, is given by

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X_\alpha(u)Y_\alpha^*(u)du \tag{5}$$

### 2.2. Wavelet transform

The continuous wavelet transform of a signal $x(t) \in L^2(\mathbb{R})$ is defined as

$$WT_x(a,b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} x(t)\varphi^*\left(\frac{t-b}{a}\right)dt \tag{6}$$