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Two and three dimensional multi-moment finite volume solver for incompressible Navier–Stokes equations on unstructured grids with arbitrary quadrilateral and hexahedral elements

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ABSTRACT

This paper presents an extension of the robust and accurate finite volume method (FVM), so-called VPM (Volume integrated average and Point value based Multi-moment) method, to structured and unstructured grids with arbitrary quadrilateral and hexahedral mesh elements. The VPM method treats two different discretized moments of the physical fields, i.e. the volume integrated average (VIA) and the point values (PV) at the vertices of each cell, as the computational variables, which distinguishes it from conventional FVM. Given the local degrees of freedom in terms of VIA and PVs, we have properly designed the interpolation polynomials of reconstruction for quadrilateral and hexahedral mesh elements, which are then used to build a numerical formulation for incompressible viscous fluid dynamics. Numerical results of benchmark tests in both 2 and 3 dimensions are presented to verify the accuracy and robustness of the proposed method, which shows significant improvement in comparison with conventional FVM. The proposed formulation provides a practical solver that is well-balanced between numerical accuracy and algorithmic complexity.

1. Introduction

The finite volume method (FVM) has become a major approach of computational fluid dynamics (CFD) for a wide spectrum of engineering applications in presence of complex geometrical configurations. In case the velocity of fluid particles is sufficiently smaller than the sound speed, the fluid can be adequately modeled by the incompressible approximation. The most widely used conventional numerical methods for incompressible flows are based on a solution procedure that directly couples the pressure with the velocity to ensure the divergence-free condition for incompressible flows. Some representatives found in the literature are the projection method [1], MAC (Marker and Cell) [2] method, SIMPLE (Semi-Implicit Method for Pressure-Linked Equations) method [3], SIMPLEC(SIMPLE Corrected) method [4], PISO(Pressure Implicit with Splitting Operators) method [5] and artificial compressibility method [6]. Improvements on these methods have been reported so far. We mention a few in [7–12] among numerous publications. More comprehensive reviews can be found in monographies on this topic, like [13,14].

Although variety of higher order spatial discretizations have been proposed, the pressure based projection approach finds its most popular implementation in conjunction with the FVM which forms the basic numerical frameworks of the most CFD codes in current use. In principle, FVM can be adopted not only to structured grids but also to unstructured grids with any type mesh cells. Regarding the spatial discretization, although the second-order FVM on unstructured meshes has been accepted to be a good trade-off between computational complexity and numerical accuracy in practical applications, some remaining problems, for example the dependency of solution quality on computational mesh and the poor accuracy in advection computation, still deserve further efforts for more accurate and robust formulations.

In the conventional FVM, where the volume average value is the only discretized computational variable that is updated in time, the numerical solution is highly dependent on the quality of computational grids, which is usually raised in terms of orthogonality, skewness and aspect ratio of the mesh cell. Another remaining issue to be addressed is that generating the interpolation reconstruction beyond the linear function for the conventional FVM in an unstructured grid is not a trivial task.

We have recently developed a finite volume formulation with improved accuracy and robustness in [15] by adding the point values (PV) at the cell vertices as another computational variable, which is a discretized moment of the physical field different from the volume integrated average (VIA). The formulation, so-called Volume integrated average and Point value based Multi-moment

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(VPM) method, improves significantly the accuracy and robustness of the numerical solution with a modest increase in the algorithmic complexity. We presented VPM method for cells of triangle and tetrahedron in [15] and showed that the VPM method is a practical formulation that well balances the solution quality and the computational cost. On other hand, it is well known that a grid with triangular and tetrahedral cells tends to degrade the solution quality and might be not adequate for applications where mesh cells of large aspect ratio are more preferable.

This paper presents a VPM scheme particularly designed for quadrilateral (2D) and hexahedral (3D) unstructured grids. It is well known that the quadrilateral/hexahedral grid allows high aspect ratio and the cells can be stretched along the main stream direction without significant loss in numerical accuracy, and thus much demanded in many applications in comparison with the triangular/tetrahedral grid. Extending the existing VPM formulation to quadrilateral/hexahedral grid, however, is not straightforward with the limited number of degrees of freedom (DOF) available for reconstruction. Special attention is needed in designing the multi-moment interpolation function.

In this paper, we present the newly designed quadratic multi-moment interpolation polynomials for quadrilateral and hexahedral cells by adding new constraint conditions in terms of the first and second derivatives of the physical field. Using this new reconstruction functions, we developed numerical solvers for incompressible Navier–Stokes equations in both two and three dimensions. Systematic benchmark tests have been carried out to verify the proposed method as a more accurate and robust option in comparison with the conventional FVM.

This paper is organized as follows. The numerical formulation is presented in Section 2. Numerical tests are given in Section 3 to verify the accuracy and the robustness of the present method in comparison with other conventional FVMs. This paper ends with some conclusion remarks in Section 4.

2. Numerical formulation on quadrilateral and hexahedral mesh cells

The target equations to be solved in this study is the incompressible unsteady Navier–Stokes equations,

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla(\mathbf{u} \otimes \mathbf{u}) = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

where $\mathbf{u} = (u, v, w)$ is the velocity vector with components u, v and w in x, y and z directions respectively. p is the pressure, ρ the density and ν the kinematic viscosity.

We use the projection method of Chorin [1] as a simple and robust solution procedure to solve Navier–Stokes equations (1) under divergence-free constraint (2) for incompressible flow. The major concern in this paper is the spatial discretization that involves both VIA and PV as the computational variables.

The underlying concept in this paper is the same as that in [15]. However, the numerical details in quadrilateral and hexahedral grids are substantially different from the triangular and tetrahedral grids. In order to make the present paper self-contained and to enable the readers to follow the algorithmic details, we present the numerical formulation in line with that in [15].

2.1. The computational mesh

In the two dimensional case, the computational domain is divided into non-overlapping convex quadrilateral cells Ω_i $(i = 1, \ldots, N_e)$ with four vertices $\theta_{ij} (j = 1, 2, 3, 4)$ located at (x_{i1}, y_{i1}), (x_{i2}, y_{i2}), (x_{i3}, y_{i3}) and (x_{i4}, y_{i4}) respectively as shown in Fig. 1. The mass center of Ω_i is denoted by θ_{ic}. The four boundary line segments of element Ω_i are denoted by $\Gamma_{i1} = \overline{\theta_{i4}\theta_{i1}}$, $\Gamma_{i2} = \overline{\theta_{i2}\theta_{i3}}$, $\Gamma_{i3} = \overline{\theta_{i1}\theta_{i2}}$ and $\Gamma_{i4} = \overline{\theta_{i3}\theta_{i4}}$. The middle points and the outward normal unit vector of segment Γ_{ij} is denoted by $\tilde{\theta}_{ij}$ and $\mathbf{n}_{ij} = (n_{xij}, n_{yij})$ respectively.

Being the computational variables, the volume-integrated average (VIA) and point-value (PV) at the cell vertices of a physical variable $\phi(x, y, t)$ are defined as:

$$\begin{aligned} &\bar{\phi}_i(t) \equiv \frac{1}{|\Omega_i|}\int_{\Omega_i} \phi(x, y, t)\, d\Omega, \\ &\phi_{ij}(t) \equiv \phi(x_{ij}, y_{ij}, t), \quad j = 1, 2, 3, 4; \end{aligned} \tag{3}$$

where $|\Omega_i|$ denotes the volume of cell element Ω_i.

For simplicity, mesh cell Ω_i is mapped onto a standard element $\omega \equiv [-1 \leqslant \xi, \eta \leq 1]$ in the local coordinate (ξ, η) where vertices $\theta_{ij}(j = 1, 2, 3, 4) = (\xi_j, \eta_j)$ correspond to $(-1,-1)$, $(-1, 1)$, $(1, 1)$ and $(1,-1)$. The transformation is given in terms of the shape functions, $\mathcal{N}_j = \frac{1}{4}(1 + \xi_j\xi)(1 + \eta_j\eta)$. So, the global coordinate can be expressed in terms of local coordinate system by

$$\begin{aligned} x &= \mathcal{N}_1 x_{i1} + \mathcal{N}_2 x_{i2} + \mathcal{N}_3 x_{i3} + \mathcal{N}_4 x_{i4}, \\ y &= \mathcal{N}_1 y_{i1} + \mathcal{N}_2 y_{i2} + \mathcal{N}_3 y_{i3} + \mathcal{N}_4 x_{i4}. \end{aligned} \tag{4}$$

It is straightforward to arrive at the formula to compute the first-order derivatives in respect to x and y in the original coordinate system by the local coordinate,

$$\begin{aligned} \phi_{xi}(\xi, \eta) &= \frac{1}{|J_i(\xi, \eta)|}\left(y_{\eta i}(\xi, \eta)\frac{\partial \phi_i(\xi, \eta)}{\partial \xi} - y_{\xi i}(\xi, \eta)\frac{\partial \phi_i(\xi, \eta)}{\partial \eta}\right), \\ \phi_{yi}(\xi, \eta) &= \frac{1}{|J_i(\xi, \eta)|}\left(-x_{\eta i}(\xi, \eta)\frac{\partial \phi_i(\xi, \eta)}{\partial \xi} + x_{\xi i}(\xi, \eta)\frac{\partial \phi_i(\xi, \eta)}{\partial \eta}\right), \end{aligned} \tag{5}$$

where $|J_i(\xi, \eta)| = x_{\xi i}(\xi, \eta)y_{\eta i}(\xi, \eta) - x_{\eta i}(\xi, \eta)y_{\xi i}(\xi, \eta)$ is the determinant of the Jacobian matrix.

The metric terms of mapping (5) are directly obtained by

$$\begin{cases} x_{\xi i} = \frac{1}{4}(-x_{i1} + x_{i2} + x_{i3} - x_{i4} + (x_{i1} - x_{i2} + x_{i3} - x_{i4})\eta) \\ x_{\eta i} = \frac{1}{4}(-x_{i1} - x_{i2} + x_{i3} + x_{i4} + (x_{i1} - x_{i2} + x_{i3} - x_{i4})\xi) \\ y_{\xi i} = \frac{1}{4}(-y_{i1} + y_{i2} + y_{i3} - y_{i4} + (y_{i1} - y_{i2} + y_{i3} - y_{i4})\eta) \\ y_{\eta i} = \frac{1}{4}(-y_{i1} - y_{i2} + y_{i3} + y_{i4} + (y_{i1} - y_{i2} + y_{i3} - y_{i4})\xi) \end{cases}, \tag{6}$$

and

$$|J_i(\xi, \eta)| = 2|\Omega_i| = x_{\xi i}y_{\eta i} - x_{\eta i}y_{\xi i}. \tag{7}$$

2.2. The multi-moment reconstruction

We use the following piecewise reconstruction polynomial for physical field $\phi(x, y)$ on cell Ω_i in the local coordinate,

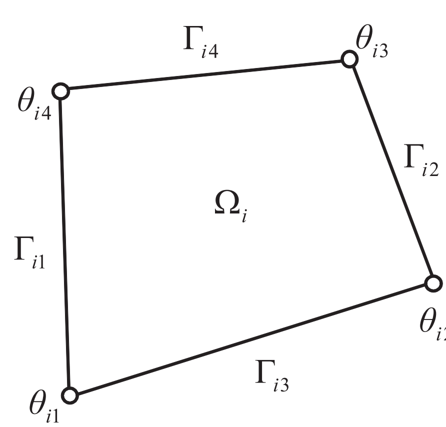

Fig. 1. The two dimensional quadrilateral mesh element.