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Global solutions for nonlinear fuzzy fractional integral and integrodifferential equations

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ABSTRACT

This paper is devoted to considering the existence and uniqueness results for fuzzy fractional integral equations employing the method of upper and lower solutions. Moreover, the approach is followed to prove the existence of solutions for the fuzzy initial value problem of fractional integrodifferential equations involving Riemann–Liouville differential operators. The method is illustrated by two examples.

1. Introduction

The theory of fractional differential equations involving Riemann–Liouville differential operators of fractional order $0 < q < 1$, has attracted more attention in the past few years because of its unique physical properties and potential for applications (see e.g. [1,2]) and therefore seems to deserve an independent study of this theory parallel to the well-known theory of ordinary differential equations. Many people have recently focused on existence results of solutions for the initial value problem of fractional differential equations [2–5].

However, to the best of our knowledge, although various results for fuzzy differential and integrodifferential equations have been established until now [6–8], results for fuzzy fractional differential equations (FFDEs) are rarely seen [9–12]. Authors in [12,10], proved the existence of a solution for the following fuzzy differential equation of fractional order $0 < q < 1$

$$\begin{aligned} &D^q u(t) = f(t,u), \quad \forall t \in J, \\ &\lim_{t\to 0^+} t^{1-q} u(t) = u_0, \end{aligned} \tag{1.1}$$

where $u_0 \in \mathbb{R}_{\mathcal{F}}, J = (0,b], f \in C(J \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}})$, and $D^q u$ denotes the fuzzy derivative of fractional order q introduced in [12,10].

The ideas in some of papers are very interesting, but we find that they are wrong about the definition of solution space for the initial value problem. The space $C([0,b],\mathbb{R}_{\mathcal{F}})$ is defined as the solution space, which is not suitable (except for $u_0 = 0$), because if $u \in C([0,b],\mathbb{R}_{\mathcal{F}})$, then $t^{1-q}u(t)|_{t=0} = 0$ (see for example [12,10]). Although the author in [12] introduced the solution space for FFDEs correctly, the integral solution is defined incorrectly (except for $u_0 = 0$) so that it affects the corresponding solution space for FFDE.

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Therefore, we first wish to investigate the existence and uniqueness results for the following fuzzy fractional integral equation

$$u(t) = g(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s), (Tu)(s)) ds, \quad t \in J,$$

where $J = (0, b], g \in C(J, \mathbb{R}_{\mathcal{F}}), f \in C(J \times \mathbb{R}^2_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}})$, and

$$(Tu)(t) = \int_0^t k(t,s) u(s) ds, \tag{1.2}$$

where $k \in C(I, \mathbb{R}_+), I = \{(t,s) \in \bar{J} \times \bar{J} : t \geqslant s\}, \mathbb{R}_+ = [0, +\infty)$ and $0 < q < 1$.

Secondly, we use these results to prove the existence theorems for the following fuzzy fractional integrodifferential equations (FFIDEs) involving Riemann–Liouville derivatives of order $0 < q < 1$. Consider

$$\begin{aligned} &D^q u(t) = f(t, u, Tu), \quad \forall\, t \in J, \\ &\lim_{t \to 0^+} t^{1-q} u(t) = u_0, \end{aligned} \tag{1.3}$$

where $u_0 \in \mathbb{R}_{\mathcal{F}}, f \in C(J \times \mathbb{R}^2_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}})$, and $D^q u$ denotes the fuzzy derivative of fractional order q introduced in [12,10].

Due to the nonlinearity f and the appearance of Tu, the integral operator, problem (1.3) is fully nonlinear fuzzy fractional integrodifferential equation and a general form of problem (1.1).

Here, our method is based on the monotone iterative technique, combined with the method of upper and lower solutions.

The importance of this method results from the fact that in using the method of upper and lower solutions, we can prove the existence and uniqueness of global solution for problem (1.3) under weak conditions on f in comparison with former works for fractional differential equations [9,11,12].

It is worth noticing that our presented method for the fuzzy problem (1.3) can be applied for fuzzy integrodifferential equation in [6] where the problem has been investigated under stronger conditions on involved functions such as bondedness and Lipschitz continuity.

2. Preliminaries

In this section, we briefly state some definitions and results related to fuzzy functions from the literature, which will be referred to throughout this paper.

$\mathbb{R}_{\mathcal{F}}$ denotes the space of fuzzy numbers on $\mathbb{R}$. For $0 < \alpha \leqslant 1$, α-level set of $x \in \mathbb{R}_{\mathcal{F}}$ is defined by $[x]^\alpha = \{t \in \mathbb{R} | x(t) \geqslant \alpha\}$ and $[x]^0 = \overline{\{t \in \mathbb{R} | x(t) > 0\}}$. For any $\alpha \in [0,1]$, $[x]^\alpha$ is a bounded closed interval, and we denote $[x]^\alpha = [x_{\alpha l}, x_{\alpha r}]$. Also, we define $\hat{0} \in \mathbb{R}_{\mathcal{F}}$ as $\hat{0}(t) = 1$ if $t = 0$, and $\hat{0}(t) = 0$ if $t \neq 0$.

If $x, y \in \mathbb{R}_{\mathcal{F}}$, and if there exists a unique fuzzy number $z \in \mathbb{R}_{\mathcal{F}}$ such that $y + z = x$, then z is called the H-difference of x, y and is denoted by $x \ominus y$ (see e.g. [13]).

We denote a triangular number as $x = (a, b, c)$, where a, c are endpoints of the 0-level set and 1-level set $= \{b\}$ (see e.g. [14,15]).

If $x, y \in \mathbb{R}_{\mathcal{F}}$, the distance between x and y is defined by

$$D(x,y) = \sup_{\alpha \in [0,1]} max\{|x_{\alpha l} - y_{\alpha l}|, |x_{\alpha r} - y_{\alpha r}|\}.$$

In this paper, for the integral concept, we will use the fuzzy Riemann integral. Also if $g : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is an integrable fuzzy function and we denote $[g(t)]^\alpha = [(g(t))_{\alpha l}, (g(t))_{\alpha r}]$, then the boundary functions $(g(t))_{\alpha l}$ and $(g(t))_{\alpha r}$ are integrable and $[\int_a^b g(t)dt]^\alpha = [\int_a^b (g(t))_{\alpha l} dt, \int_a^b (g(t))_{\alpha r} dt], \alpha \in [0,1]$. Let $g \in C((0,b], \mathbb{R}_{\mathcal{F}})$. We say that $g \in L^1((0,b], \mathbb{R}_{\mathcal{F}})$ if and only if $D(\int_0^b g(s)ds, \hat{0}) < \infty$.

Definition 2.1 (*See e.g. [16]*). Let $g : (a,b) \to \mathbb{R}_{\mathcal{F}}$ and $t_0 \in (a,b)$. We say g is generalized differentiable at t_0, if there exists an element $g'(t_0) \in \mathbb{R}_{\mathcal{F}}$, such that

(i) for all $h > 0$ sufficiently small, there exist $g(t_0 + h) \ominus g(t_0)$, $g(t_0) \ominus g(t_0 - h)$ and the limits (in the metric D)

$$\lim_{h \searrow 0} \frac{g(t_0 + h) \ominus g(t_0)}{h} = \lim_{h \searrow 0} \frac{g(t_0) \ominus g(t_0 - h)}{h} = g'(t_0),$$

or

(ii) for all $h > 0$ sufficiently small, there exist $g(t_0) \ominus g(t_0 + h)$, $g(t_0 - h) \ominus g(t_0)$ and the limits

$$\lim_{h \searrow 0} \frac{g(t_0) \ominus g(t_0 + h)}{-h} = \lim_{h \searrow 0} \frac{g(t_0 - h) \ominus g(t_0)}{-h} = g'(t_0),$$

(h and $-h$ at denominators mean $\frac{1}{h}\cdot$ and $-\frac{1}{h}\cdot$, respectively).