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ORIGINAL ARTICLE

The exact solution of a class of boundary value problems with polynomial coefficients and its applications on nanofluids

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Abstract Usually, the temperature distribution of nanofluids and the nanoparticles' concentration are finally governed by second-order ordinary differential equations with polynomial coefficients. In this work, a class of second-order boundary value problems with applications on nanofluids has been theoretically solved in terms of the Kummer function. Several lemmas have been presented to relate the Kummer function with the generalized incomplete gamma function. Accordingly, the current solutions reduce to those in the literature at certain values of the coefficients as special cases. Furthermore, the present results are very useful in obtaining the solutions for any future similar problems without any need to perform further calculations.

1. Introduction

Recently, it has been found that the temperature distribution and the nanoparticles' concentration of nanofluids (Aly et al., 2013a,b, 2016; Ebaid et al., 2014; Ebaid and Al Sharif, 2015; Hamad, 2011; Kameswaran et al., 2012; Khan et al., 2013) are basically governed by partial differential equations. These basic equations are then reduced to a set of ordinary differential equations by means of a similarity variable. The final transformed ODEs are usually of second order with polynomial coefficients. In this paper, we consider a generalized second-order ordinary differential equation describing the temperature of nanofluids and/or the nanoparticles concentration $z(t)$ in the form:

$$tz''(t) + (P + Qt)z'(t) + Rz(t) = 0, \tag{1}$$

subject to the following set of boundary conditions

$$z(0) = 0, \quad z(1) = 1 + \xi z'(1), \quad \xi \in \Re, \tag{2}$$

where P, Q, and R are physical parameters which are related to the densities, the thermal conductivities, and the heat capacitances of the base-fluids and the nanofluids. The parameter ξ is often used to describe the convective heat condition and it takes some particular values according to the physical problem. In this filed of research, many useful results have been recently reported by Bhatti et al. (2016, 2017), Ellahi et al. (2016), Shehzad et al. (2016), Sheikholeslami and Zeeshan (2017a,b), andZeeshan et al. (2016). The objective of this paper is to derive a general exact solution for the system (1)–(2). The paper is organized as follows. In Section 2, the

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solution of Eqs. (1)–(2) will be obtained in terms of hypergeometric function. In Section 3, some special cases are discussed. Section 4 is devoted to obtain some useful properties through some lemmas. The solution in terms of the generalized incomplete gamma function is introduced in Section 5. Moreover, it will be proved in Section 6 that the current general solution reduces to the those in the literature as special cases.

2. Solution in terms of hypergeometric function

Following Ebaid et al. (2017), the analytical solution of Eqs. (1)–(2) can be obtained as

$$z(t) = \frac{ct^{1-P}}{\Gamma(2-P)}\,{}_1F_1\left[1-P+\frac{R}{Q}, 2-P, -Qt\right], \quad P<1, \tag{3}$$

where c is an integration constant and ${}_1F_1$ is Kummer's function. In addition, the assumption $P<1$ in (3) leads to the satisfaction of the first boundary condition ($z(0)=0$), while the second boundary condition $z(1)=1+\xi z'(1)$ determines c as

$$c = \frac{(2-P)\Gamma(2-P)}{(2-P)[1-\xi(1-P)]\,{}_1F_1\left[1-P+\frac{R}{Q},2-P,-Q\right]+\xi((1-P)Q+R)\,{}_1F_1\left[2-P+\frac{R}{Q},3-P,-Q\right]}. \tag{4}$$

Therefore $z(t)$ is finally given by

$$z(t) = \frac{(2-P)t^{1-P}\,{}_1F_1\left[1-P+\frac{R}{Q},2-P,-Qt\right]}{(2-P)[1-\xi(1-P)]\,{}_1F_1\left[1-P+\frac{R}{Q},2-P,-Q\right]+\xi((1-P)Q+R)\,{}_1F_1\left[2-P+\frac{R}{Q},3-P,-Q\right]}. \tag{5}$$

It will be shown in a later section that the general solution (5) reduces to the same results in the literature as special cases. However, the limitation of the current method can be found if Eq. (1) is non-homogenous, i.e., the R.H.S. of this equation contains some terms in t.

3. Special cases

3.1. $R\to 0, \xi\nrightarrow 0$

When $R\to 0$, Eq. (1) becomes

$$tz''(t)+(P+Qt)z'(t)=0, \tag{6}$$

which arises in Hoda et al. (2017). Hence, the exact solution in this case can be directly obtained from (5) as

$$z(t) = \frac{(2-P)t^{1-P}\,{}_1F_1[1-P,2-P,-Qt]}{(2-P)[1-\xi(1-P)]\,{}_1F_1[1-P,2-P,-Q]+\xi Q(1-P)\,{}_1F_1[2-P,3-P,-Q]}. \tag{7}$$

3.2. $R\nrightarrow 0, \xi\to 0$

When ξ vanishes, the solution directly comes from (7) as

$$z(t) = \frac{t^{1-P}\,{}_1F_1\left[1-P+\frac{R}{Q},2-P,-Qt\right]}{{}_1F_1\left[1-P+\frac{R}{Q},2-P,-Q\right]}. \tag{8}$$

3.3. $R\to 0, \xi\to 0$

When both of R and ξ vanish, the solution is obtained from (7) or (8) as

$$z(t) = \frac{t^{1-P}\,{}_1F_1[1-P,2-P,-Qt]}{{}_1F_1[1-P,2-P,-Q]}. \tag{9}$$

It is important here to refer to that the solutions (7) and (9) can be expressed in terms of the generalized incomplete gamma function. In order to do that, some relations between the Kummer function and the generalized incomplete gamma function shall be proved in the following section.

4. Analysis

4.1. Theorem 1

For $b=a+1$, we have

$${}_1F_1[a,b,\tau] = a(-\tau)^{-a}\Gamma(a,0,-\tau). \tag{10}$$