ARTICLE IN PRESS

Journal of the Association of Arab Universities for Basic and Applied Sciences (2016) **xxx**, xxx–xxx

University of Bahrain

Journal of the Association of Arab Universities for Basic and Applied Sciences

www.elsevier.com/locate/jaaubas
www.sciencedirect.com

An efficient numerical method for the modified regularized long wave equation using Fourier spectral method

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Received 4 February 2016; revised 21 August 2016; accepted 8 October 2016

KEYWORDS

MRLW equation;
Fourier spectral method;
Fast Fourier transform;
Leap-Frog method;
Solitary waves

Abstract The modified regularized long wave (MRLW) equation is numerically solved using Fourier spectral collection method. The MRLW equation is discretized in space variable by the Fourier spectral method and Leap-Frog method for time dependence. To validate the efficiency, accuracy and simplicity of the used method, four cases study are solved. The single soliton wave motion, interaction of two solitary waves, interaction of three solitary waves and a Maxwellian initial condition pulse are studied. The L_2 and L_∞ error norms are computed for the motion of single solitary waves. To determine the conservation properties of the MRLW equation three invariants of motion are evaluated for all test problems.

1. Introduction

The regularized long wave (RLW) equation

$$u_t + u_x + uu_x - \mu u_{xxt} = 0, \tag{1.1}$$

where μ is a positive constant, is a nonlinear evolution equation, which was originally introduced by Peregrine (1966) in describing the behavior of an undular bore and studied later by Benjamin et al. (1972). This equation plays an important role in describing physical phenomena in various disciplines, such as the nonlinear transverse waves in shallow water, ion-acoustic waves in plasma, magneto–hydrodynamics waves in plasma, longitudinal dispersive waves in elastic rods, and pressure waves in liquid's gas bubbles. Many numerical methods for the RLW equation have been proposed, such as the finite element method, Galerkin method, collocation methods with quadratic B-splines, an explicit multistep method, finite difference methods and Fourier Leap-Frog method (Liu et al., 2013; Saka and Dag, 2008; Soliman and Raslan, 2001; Mei and Chen, 2012; Lin et al., 2007; Hassan and Saleh, 2010). The RLW equation is a special case of the generalized regularized long wave (GRLW) equation

$$u_t + u_x + \delta u^p u_x - \mu u_{xxt} = 0, \tag{1.2}$$

where δ and μ are positive constants and p is a positive integer. Various numerical techniques have been used for the solution of the GRLW equation as (Mohammadi and Mokhtari, 2011; Kaya, 2004; Roshan, 2012; Hammad and El-Azab, 2015; Zeybek and Karakoç, 2016). The modified regularized long wave equation (MRLW) is a special form of GRLW Eq. (1.2) and it plays a very important role at the modeling of the nonlinear, dispersive media being modeled feature

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Peer review under responsibility of University of Bahrain.

http://dx.doi.org/10.1016/j.jaubas.2016.10.002
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small-amplitude, long-wave length disturbances. The MRLW equation was also solved using various numerical methods such as, a Galerkin finite element method, a spline method, the Adomian decomposition method, a collocation method with cubic B-splines, finite difference scheme, meshless kernel based method of lines, B-spline finite elements, mixed Galerkin finite element methods, Tri-prong scheme, homotopy perturbation method and He's variational iteration method as Mei et al. (2014), Raslan and EL-Danaf (2010), Raslan and Hassan (2009), Khalifa et al. (2007, 2008a,b), Dereli (2012), Gardner et al. (1997), Gao and Mei (2015), Hosseini et al. (2016), Achouri and Omrani (2010), Labidi and Omrani (2011). Discretization using finite differences in time and spectral methods in space has proved to be efficient in solving numerically nonlinear partial differential equations (PDE) describing wave propagation. The combined schemes have been applied efficiently to analyze unidirectional solitary wave propagation in one dimension Korteweg de Vries (KdV) equation as Fornberg (1996), Fornberg and Whitham (1987), Hassan and Saleh (2013). The combination of spectral methods and finite differences is applied to the Boussinesq type which admits bidirectional wave propagation as Hassan (2010), Borluk and Muslu (2015). The numerical solution for the modified equal width wave (MEW) equation is presented using Fourier spectral method by Hassan (2016). Different analytical and numerical methods are used to solve differential equations as Atangana and Cloot (2013), Atangana (2016), Semary and Hassan (2016), El-Borai et al. (2017). In this study, the combination of Fourier spectral method in space and leap frog in time is applied to the modified regularized long wave equation (MRLW) equation. Consider the MRLW equation

$$u_t + u_x + 6u^2u_x - \mu u_{xxt} = 0, \tag{1.3}$$

where the subscripts x and t denote differentiation, is considered with the boundary conditions $u \to 0$ as $x \to \pm\infty$. In this study, boundary conditions are chosen from

$$u(a,t) = 0, \quad u(b,0) = 0, \quad t > 0. \tag{1.4}$$

and the initial condition

$$u(x,0) = f(x), \quad a \leqslant x \leqslant b. \tag{1.5}$$

where function $f(x)$ will be chosen later. The numerical solution of the MRLW equation is investigated using the Fourier Leap-Frog methods. The used method is validated by studying the motion of a single solitary wave, development of interaction of two positive solitary waves, development of three positive solitary waves interaction and a Maxwellian initial condition pulse is then studied.

2. Analysis of the numerical scheme

A numerical method is developed for the periodic initial value problem in which u is a prescribed function of x at $t = 0$ and the solution is periodic in x outside a basic interval $a \leqslant x \leqslant b$. Interval may be chosen large enough so the boundaries do not affect the propagation of solitary waves. The Eq. (1.1) can be written as

$$w_t = -u_x - 6u^2u_x \tag{2.6}$$

where

$$w = u - \mu u_{xx} \tag{2.7}$$

For ease of presentation the spatial period $[a, b]$ is normalized to $[0, 2\pi]$ using the transformation $x \to 2\pi(x-a)/L$, where $L = b - a$. $u(x,t)$ is transformed into Fourier space with respect to x, and derivatives (or other operators) with respect to x. This operation can be done with the Fast Fourier transform (FFT). Applying the inverse Fourier transform $\frac{\partial^n u}{\partial x^n} = F^{-1}(ik)^n F(u), n = 1, 2, \ldots$. Then, we need to discretize the results equations. For any integer $N > 0$ consider $x_j = j\Delta x = \frac{2\pi j}{N}, j = 0, 1, \ldots, N-1$. The solution $u(x,t)$ is transformed into the discrete Fourier space as

$$\hat{u}(k,t) = F(u) = \frac{1}{N}\sum_{j=0}^{N-1} u(x_j,t)e^{-ikx_j}, \quad -\frac{N}{2} \leqslant k \leqslant \frac{N}{2} - 1 \tag{2.8}$$

And the inverse formula is

$$u(x_j,t) = F^{-1}(\hat{u}) = \sum_{k=-N/2}^{N/2-1} \hat{u}(k,t)e^{ikx_j}, \quad 0 \leqslant j \leqslant N-1 \tag{2.9}$$

After all the previous mathematical operations to Eqs. (2.7) and (2.6), and then reducing the resulting equation to the equations

$$w(x_j,t) = u(x_j,t) - \mu(2\pi/L)^2 F^{-1}\{-k^2F(u)\}, \tag{2.10}$$

$$\frac{\partial w(x_j,t)}{\partial t} = -(2\pi/L)F^{-1}\{ikF(u)\} - 6(2\pi/L)^2u^2(x_j,t)F^{-1}\{ikF(u)\}. \tag{2.11}$$

Letting $\boldsymbol{u} = [u(x_0,t), u(x_1,t), \ldots, u(x_{N-1},t)]^T$.

The ordinary differential equation (2.11) can be written in the vector form

$$\boldsymbol{w_t} = g(\boldsymbol{u}) \tag{2.12}$$

where $g(\boldsymbol{u})$ defines the right hand side of (2.11). The Leap Frog method (two-step scheme) is given as

$$w_t = \frac{w(x,t+\Delta t) - w(x,t-\Delta t)}{2\Delta t} = \frac{w^{n+1} - w^{n-1}}{2\Delta t} \tag{2.13}$$

is used to solve the resulting ordinary differential equation (2.12) in time. Use the Leap-Frog scheme to advance in time to obtain $w(x,t+\Delta t) = w(x,t-\Delta t) + 2\Delta t g(u(x,t))$.

Finally, we find the approximate solution using the inverse Fourier transform (2.9). The Leap-Frog needs two levels of initial value; we begin with $u(x,0)$ to get $w(x,0)$ from (2.10), then

$$w(x,n\Delta t) = F^{-1}((1+\mu k^2(2\pi/L)^2)F(u(x,n\Delta t)) \tag{2.14}$$

$$w(x,0) = F^{-1}((1+\mu k^2(2\pi/L)^2)F(u(x,0))). \tag{2.15}$$

Then evaluate the second level of initial solution $w(x,\Delta t)$ by using a higher-order one-step method, for example, a fourth-order Runge–Kutta method (RK4), then substitute $w(x,\Delta t)$ in (2.14) as

$$u(x,n\Delta t) = F^{-1}(F(w(x,\Delta t)/(1+\mu k^2(2\pi/L)^2))). \tag{2.16}$$

to obtain $u(x,t)$. Thus, Eq. (2.12) become

$$w(x,t+\Delta t) = w(x,t-\Delta t) - 2\Delta t(1 + 6(2\pi/L)u^2(x,t))F^{-1}\{ikF\{u(x,t)\}\} \tag{2.17}$$

By substituting $w(x,0)$ and $u(x,\Delta t)$ in (2.17) to evaluate $w(x,2\Delta t)$ then substitute $w(x,2\Delta t)$ in (2.16) to evaluate