Journal of the Association of Arab Universities for Basic and Applied Sciences (2016) xxx, xxx–xxx

University of Bahrain
Journal of the Association of Arab Universities for Basic and Applied Sciences
www.elsevier.com/locate/jaaubas
www.sciencedirect.com

A note on two variable Laguerre matrix polynomials

Maged G. Bin-Saad

Department of Mathematics, Faculty of Science, Aden University, Aden, Yemen

Received 22 November 2015; revised 31 July 2016; accepted 21 September 2016

KEYWORDS
Laguerre matrix polynomials;
Operational images;
Functional relations;
Differential matrix equations

Abstract The principal object of this paper is to present a natural further step toward the mathematical properties and presentations concerning the two variable Laguerre matrix polynomials defined in (Bin-Saad, Maged G., Antar, A. Al-Sayaad, 2015. Study of two variable Laguerre polynomials via symbolic operational images. Asian J. of math. and comput. research, 2(1), 42–50). Series expansions, integral transforms and bilinear and bilateral generating matrix functions for these polynomials are established. Some particular cases and consequences of our main results are also considered.

1. Introduction

The subject of Laguerre polynomials has gained importance during the last two decades mainly due to its applications in various fields of mathematical physics, such as the solving of delay differential equations (Suayip et al., 2014), pantograph-type Volterra integro-differential equations (Suayip, 2014) and fractional differential equations (Bhrawy and Alghamdi, 2012; Bhrawy et al., 2014a,b; Bhrawy et al., 2015b,c). Numerous other works dealing also with the use of Laguerre polynomials and matrices include those by Ahmadian et al. (2015) in the theory of Tau method for numerical solution of fuzzy fractional kinetic model, by Bhrawy and Taha (2012) and Bhrawy et al. (2015a) in the theory of operational matrix of fractional integration of Laguerre polynomials and generalized Laguerr e–Gauss–Radau schema for first order hyperbolic equation on semi-finite domain (Abdelkawy and Taha, 2012) and so on (see also Bin-Saad and Antar (2015)). Further, matrix polynomials seen in the study of many area such as statistics, Lie group theory and number theory are well known. Recently, the matrix versions of the classical families orthogonal polynomials such as Laguerre, Jacobi, Hermite, Gegenbauer, Bessel and Humbert polynomials and some other polynomials were introduced by many authors for matrices in $\mathbb{C}^{N\times N}$ and various properties satisfied by them were given from the scalar case, see for example (Aktas et al., 2013; Aktas et al., 2011; Altin and Cekim, 2012a,b, 2013; Bin-Saad and Antar, 2015; Cekim and Erkus-Duman, 2014; Jódar and Cortés, 1998a,b; Jódar and Company, 1996; Jódar et al., 1995; Pathan et al., 2014; Bayram and Altin, 2015).

If $A_0, A_1, \ldots, A_n \ldots$, are elements of $\mathbb{C}^{N\times N}$ and $A_n \neq 0$, then we call

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \cdots + A_1 x + A_0,$$

a matrix polynomial of degree n in x. If $A + nI$ is invertible for every integer $n \geqslant 0$ then.

$$(A)_n = A(A+I)(A+2I)\cdots(A+(n-1)I);\ n \geqslant 1;\ (A)_0 = I. \tag{1.1}$$

E-mail address: mgbinsaad@yahoo.com
Peer review under responsibility of University of Bahrain.

http://dx.doi.org/10.1016/j.jaubas.2016.09.001
1815-3852 © 2016 University of Bahrain. Publishing services by Elsevier B.V.

In Bin-Saad and Antar (2015), it is shown that an appropriate combination of methods, relevant to operational calculus and to matrix polynomials, can be a very useful tool to establish and treat a new class of two variable Laguerre matrix polynomials in the following form

$$L_{n,m}^{(A,\lambda)}(x,y)=\sum_{s=0}^{n}\sum_{k=0}^{m}\frac{(-1)^{s+k}(\lambda x)^{s}(\lambda y)^{k}}{s!k!(n-s)!(m-k)!}(A+I)_{n+m}\times\left[(A+I)_{s+k}\right]^{-1},\ \{n,m\}\geqslant 0, \tag{1.2}$$

where A be a matrix in $\mathbb{C}^{N\times N}$ where $(-\alpha)$ is not an eigenvalue of A for every integer $\alpha>0$ and λ be a complex number whose real part is positive. The authors in Bin-Saad and Antar (2015) explored the formal properties of the operational identities to derive a number of properties of the new class two variable Laguerre matrix polynomials (1.2) and discussed the links with various known polynomials. The generating relation for the matrix function $L_{n,m}^{(A,\lambda)}(x,y)$, is given by the following formula (Bin-Saad and Antar, 2015):

$$[(1-u-v)^{-(A+I)}exp\left[\frac{-\lambda(xu+yv)}{1-u-v}\right]=\sum_{n,m=0}^{\infty}L_{n,m}^{(A,\lambda)}(x,y)u^{n}v^{m}, \tag{1.3}$$

where $\{u,v,x,y\}\in\mathbb{C}$ and $|u+v|<1$.

By setting $m=y=0$ in (1.2), Eq. (1.2) immediately yields the following Laguerre matrix polynomials due to Jódar and Cortés (1998a,b):

$$L_{n}^{(A,\lambda)}(x)=\sum_{s=0}^{n}\frac{(-1)^{s}}{s!(n-s)!}(A+I)_{n}[(A+I)_{s}]^{-1}(\lambda x)^{s},\ \lambda\geqslant 0. \tag{1.4}$$

For the purpose of this work, we recall here same definitions.

Definition 1.1. Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$, then Gamma matrix function is defined by (Jódar and Cortés, 1998a,b)

$$\Gamma(A)=\int_{0}^{\infty}e^{-1}t^{A-I}dt. \tag{1.5}$$

Definition 1.2. Let A, B and $A+B$ be a positive stable matrix in $\mathbb{C}^{N\times N}$ and $AB=BA$, then Beta matrix function is defined by (Jódar and Cortés, 1998a,b)

$$\beta(A,B)=\Gamma(A)\Gamma^{-1}(B)\Gamma(A+B)=\int_{0}^{1}t^{A-I}(1-t)^{B-I}dt. \tag{1.6}$$

Lemma 1.1. *For matrix $A(k,n)$ in $\mathbb{C}^{N\times N}$ where $n\geqslant 0,\ k\geqslant 0$, we have (see Srivastava and Manocha (1984)):*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(n,k)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(n,n-k), \tag{1.7}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(n,k)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(n,n+k). \tag{1.8}$$

Motivated by the important role of the Laguerre matrix polynomials in several diverse fields of physics and the contributions in Bayram and Altin (2013) and Jódar et al. (1994) toward the generalization of the Laguerre polynomials, this work aims at investigating several properties for the two variable Laguerre matrix polynomials $L_{n,m}^{(A,\lambda)}(x,y)$. We establish some projection series, integral transforms and bilinear and bilateral generating matrix functions. Many earlier (known) results given by Bayram and Altin (2013) are shown to be special cases of our results.

2. Finite and infinite sums

Theorem 2.1. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ satisfying spectral condition $(-\alpha)$ is not an eigenvalue of A for every integer $\alpha>0$, $(-\beta)$ is not an eigenvalue of B for every integer $\beta>0$, $AB=BA$, $\Re(\lambda)>0$ and $n\geqslant 0$, $m\geqslant 0$. Then*

$$\sum_{s=0}^{n}\sum_{k=0}^{m}\frac{(A+B)_{n+m-s-k}}{(n-s)!(m-k)!}L_{s,k}^{(B,\lambda)}(x,y)=L_{n,m}^{(A,\lambda)}(x,y). \tag{2.1}$$

Proof. Using (1.8), we can write

$$\sum_{n,m=0}^{\infty}\sum_{s=0}^{n}\sum_{k=0}^{m}\frac{(A+B)_{n+m-s-k}}{(n-s)!(m-k)!}L_{s,k}^{(B,\lambda)}(x,y)u^{n}v^{m}=\sum_{n,m,s,k=0}^{m}\frac{(A+B)_{n+m}}{n!m!}L_{s,k}^{(B,\lambda)}(x,y)u^{n+s}v^{m+k},$$

which on using the multinomial formula

$$(1-x-y)^{-\lambda}=\sum_{n,m=0}^{\infty}\frac{(\lambda)_{n+m}x^{n}y^{m}}{n!m!},$$

gives us

$$\sum_{n,m=0}^{\infty}\sum_{s=0}^{n}\sum_{k=0}^{m}\frac{(A+B)_{n+m-s-k}}{(n-s)!(m-k)!}L_{s,k}^{(B,\lambda)}(x,y)u^{n}v^{m}=(1-u-v)^{-(A-B)}\sum_{s,k=0}^{\infty}L_{s,k}^{(B,\lambda)}(x,y)u^{s}v^{k}. \tag{2.2}$$

Now, employing (1.3) in (2.2) and comparing the coefficients of $u^{n}v^{m}$ in the resulting expression, we get the desired result. □

Theorem 2.2. *Let A be matrix in $\mathbb{C}^{N\times N}$ satisfying spectral condition $(-\alpha)$ is not an eigenvalue of A for every integer $\alpha>0$, $\Re(\lambda)>0$, $\{|z|,|w|\}<1$ $n\geqslant 0$, $m\geqslant 0$. Then*

$$\sum_{s=0}^{n}\sum_{k=0}^{m}\frac{(A+(s+k+1)I)_{n+m-s-k}(1-z)^{n}(1-w)^{m}}{(n-s)!(m-k)!}L_{s,k}^{(A,\lambda)}(x,y)\left(\frac{z}{1-z}\right)^{s}\left(\frac{w}{1-w}\right)^{k}=L_{n,m}^{(A,\lambda)}(xz,yw), \tag{2.3}$$

Proof. According to formula (1.8), we can write

Please cite this article in press as: Bin-Saad, M.G. A note on two variable Laguerre matrix polynomials. Journal of the Association of Arab Universities for Basic and Applied Sciences (2016), http://dx.doi.org/10.1016/j.jaubas.2016.09.001