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A general solution for plane problem of anisotropic media containing elliptic inhomogeneity with polynomial eigenstrains

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ABSTRACT

A general complex function method is proposed to solve the plane problem for a single anisotropic elliptic inhomogeneity embedded in an infinite anisotropic medium. The system is subjected to polynomial eigenstrains as well as far-field stresses. A general procedure based on Laurent series is presented using continuous conditions at the interface. Numerical examples are given and distribution of stresses and displacements at the interface e are analyzed for prescribed polynomial eigenstrains of degrees 0, 1 and 2. Effect of inclined angle of principal axes for anisotropic material on translation and rotation of the inhomogeneity is also illustrated. For a circular inhomogeneity, its anisotropy may cause asymmetrical deformation under uniform eigenstrains.

1. Introduction

Eshelby [1–3] pointed out that both internal and external elastic fields of an ellipsoidal inhomogeneity (or inclusion) embedded in an infinite isotropic elastic medium (or matrix) could be expressed by eigenstrains, and conjectured that only ellipsoidal inhomogeneity has uniform elastic field induced by uniform eigenstrains inside the inhomogeneity. Subsequently, many researchers [4–6] studied the system of the matrix/inhomogeneity systematically.

As most engineering materials contain different shapes of inhomogeneities or inclusions, many researchers have carried out studies of Eshelby's property on inhomogeneities. Mura [7,8] studied the problem of an m-pointed polygonal inclusion and concluded that the stress field is uniform if m is odd for the inclusion induced by uniform eigenstrains. Rodin [9] provided a proof that the stress field is non-uniform for polygonal and polyhedral inclusions. Markenscoff [10] showed that polyhedral shapes are impossible for inclusion with constant eigenstresses. Recently, Zou et al. [11] showed that the Eshelby tensor field inside a non-elliptical inclusion is quite non-uniform.

In the study of anisotropic medium containing inhomogeneities, Stroh's formulation based on 2-D anisotropic elastic mechanics has been widely used. By means of Stroh's formulation and Green's function, Bacon et al. [12] studied anisotropic medium containing defects, such as dislocations, inclusions and point defects. Ting and others [13–15] studied the problem for anisotropic medium containing elliptic inclusions with perfect interface and elliptic holes, and obtained the real form of elastic stress field in the inclusion and stress concentration factor for the hole. Using the method of conformal mapping, Ru [16] analyzed Eshelby's problem of an inclusion of arbitrary shape within an anisotropic plane or half-plane of the same elastic constants. Pan [17] presented an exact closed-form solution for the Eshelby problem of polygonal inclusion in anisotropic piezoelectric full- and half-planes using line integral on the boundary of the inclusion with the integrand being the Green's function. Recently, Xu et al. [18] provided a proof of Eshelby's property for anisotropic inclusions with perfect or dislocation-like interface in plane and anti-plane problems.

Apart from Eshelby's study on ellipsoidal inclusions subject to the uniform eigenstrains, Asaro and Barnett [19], using elastic Green's functions for a general infinite anisotropic medium, showed that when an anisotropic ellipsoidal inclusion embedded in the medium undergoes eigenstrains, expressed as a polynomial of degree *M*, the final stress and strain state in the transformed inclusion is also a polynomial of degree *M*. Kinoshita [20] and Mura [21] studied the problem for infinite medium containing an anisotropic ellipsoidal inclusion subjected to polynomial eigenstrains. Mura and Kinoshita analyzed elastic field and displacement in the inclusion, and further developed the result of Asaro and Barnett. Rahman [22] confirmed Eshelby's property with his work on isotropic elliptical inclusion subjected to polynomial eigenstrains. More recently, Liu [23] provide a proof of polynomial

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eigenstress inducing polynomial strain of the same degree in an ellipsoidal inclusion. Chen [24,25] obtained closed-form solutions for Eshelby's elliptic inclusions in isotropic medium about the anti-plane and plane problem by means of complex analysis.

In the theory of elasticity, variational principle within the calculus of variations is commonly used. In the early 1960s, Jaswon and Bhargava [26] obtained elastic field for an elliptic inclusion using minimum potential energy and functions of complex variables. Bhargava and Radhakrishna [27–29] as well as Roy et al. [30,31] obtained the elastic field for isotropic and orthotropic anisotropic media containing an elliptic inclusion/inhomogeneity. Assuming that elastic field in inclusions is in the form of polynomial when eigenstrains are polynomials, Nie et al. [32–34] obtained analytical solutions for orthogonal anisotropic medium containing an elliptic inclusion.

The study of solutions of anisotropic Eshelby's inhomogeneity with non-uniform eigenstrains and its property are of great importance in understanding the mechanism of strength and failure of such heterogeneous anisotropic materials. Using complex representative theory of Lekhnitskii [35], this paper provides a concise method to obtain solutions for the problem of anisotropic media containing an elliptic inhomogeneity. The complex functions are expanded in terms of Laurent series in matrix and the inhomogeneity. According to the continuous conditions of the stresses and displacements at the interface in the physical plane, the sets of algebraic equations for the unknown coefficients can be obtained and evaluated for a prescribed form of polynomial eigenstrains.

The resulting solution for the 2D elastic field in the system is valid for the case of arbitrary anisotropy for both inhomogeneity and matrix materials as well as for the case of arbitrary location of elliptic inhomogeneity in the matrix. Numerical examples are given to illustrate the stresses and the deformation at the interface, including translation and rotation for the inhomogeneity for polynomial eigenstrains of degrees 0, 1 and 2. Effect of inclined angle of the principal axes for anisotropic material on translation and rotation of the inhomogeneity is discussed. The method and procedure proposed in this paper can be readily used to evaluate the strength and deformation of such heterogeneous anisotropic materials under polynomial eigenstrains.

2. Mathematical formulation

The equilibrium equations for the plane problem are expressed as follows:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \tag{1}$$

where σ_x, σ_y and τ_{xy} are components of stresses. Introducing Airy stress function, $F(x, y)$, as follows:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}, \tag{2}$$

The equilibrium Eq. (1) is then satisfied. The compatibility equation is given by

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \tag{3}$$

where ε_x, ε_y and γ_{xy} are strain components.

The constitutive relations under the plane stress condition can be written as follows:

$$\begin{aligned} \varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\tau_{xy}, \\ \varepsilon_y &= a_{21}\sigma_x + a_{22}\sigma_y + a_{26}\tau_{xy}, \\ \gamma_{xy} &= a_{61}\sigma_x + a_{62}\sigma_y + a_{66}\tau_{xy}, \end{aligned} \tag{4}$$

where $a_{ij} = a_{ji}$ $(i = 1, 2, 6, j = 1, 2, 6)$ are compliance elements and can be expressed in terms of engineering material constants such that

$$a_{11} = \frac{1}{E_x}, \quad a_{22} = \frac{1}{E_y}, \quad a_{21} = a_{12} = -\frac{\nu_{xy}}{E_x} = -\frac{\nu_{yx}}{E_y},$$

$$a_{66} = \frac{1}{G_{xy}}, \quad a_{61} = a_{16} = \frac{\eta_{xy,x}}{E_x}, \quad a_{62} = a_{26} = \frac{\eta_{xy,y}}{E_y},$$

in which E_x and E_y are two elastic moduli in the x- and y-directions, respectively. ν_{xy} is Poisson's ratio, G_{xy} is shear modulus in the xy coordinate plane, $\eta_{xy,x}$ and $\eta_{xy,y}$, are mutual influence coefficients characterizing respectively extensions in the x- and y-directions of the coordinate axes due to the shear stress in the xy plane. When the principal axes of the orthotropic materials coincide with the Cartesian coordinate axes, $\eta_{xy,x} = \eta_{xy,y} = 0$.

Substitution of Eqs. (2) and (4) into Eq. (3) leads to a compatibility equation in terms of the stress function

$$a_{22}\frac{\partial^4 F}{\partial x^4} - 2a_{26}\frac{\partial^4 F}{\partial x^3 \partial y} + (2a_{12} + a_{66})\frac{\partial^4 F}{\partial x^2 \partial y^2} - 2a_{16}\frac{\partial^4 F}{\partial x \partial y^3} + a_{11}\frac{\partial^4 F}{\partial y^4} = 0. \tag{5}$$

Solution for Eq. (5) has the form of

$$F(x, y) = \sum_{k=1}^{4} F_k(x + \mu_k y), \tag{6}$$

where $\mu_i, i = 1, \ldots, 4$, are four roots of the resulting characteristic equation

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0. \tag{7}$$

For ideal elastic materials, the four roots correspond to two pairs of complex conjugates, i.e., $\mu_3 = \overline{\mu}_1, \mu_4 = \overline{\mu}_2$, and $\mu_k = \alpha_k + i\beta_k$, $(\beta_k > 0, k = 1, 2)$ where α_k and β_k are real and imaginary parts, respectively. $\mu_k, k = 1, 2$ are two basic complex parameters characterizing the degree of anisotropy. Eq. (6) can thus be written as follows:

$$F = 2\mathrm{Re}\sum_{k=1}^{2} F_k(z_k) \tag{8}$$

where

$$z_k = x + \mu_k y, \quad k = 1, 2, \tag{9}$$

indicate two physical complex planes for anisotropic materials. Introducing two generalized stress functions such that

$$\varphi_k(z_k) = \frac{dF_k}{dz_k}, \varphi_k'(z_k) = \frac{d\varphi_k}{dz_k}, \quad k = 1, 2, \tag{10}$$

then

$$\frac{\partial F}{\partial x} = 2\mathrm{Re}\sum_{k=1}^{2} \varphi_k(z_k), \frac{\partial F}{\partial y} = 2\mathrm{Re}\sum_{k=1}^{2} \mu_k\varphi_k(z_k). \tag{11}$$

The stress and displacement components can be expressed as follows:

$$\begin{aligned} \sigma_x(x, y) &= 2\mathrm{Re}\sum_{k=1}^{2} \mu_k^2\varphi_k'(z_k), \\ \sigma_y(x, y) &= 2\mathrm{Re}\sum_{k=1}^{2} \varphi_k'(z_k), \\ \tau_{xy}(x, y) &= -2\mathrm{Re}\sum_{k=1}^{2} \mu_k\varphi_k'(z_k), \end{aligned} \tag{12}$$

and

$$u(x, y) = 2\mathrm{Re}\sum_{k=1}^{2} p_k\varphi_k(z_k),$$