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Axisymmetric rotational stagnation point flow impinging on a radially stretching sheet

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ABSTRACT

The normal impingement of the rotational stagnation-point flow of Agrawal (1957) [8] on a sheet radially stretching at non-dimensional stretch rate β is studied. A similarity reduction of the Navier–Stokes equations yields an ordinary differential equation which is solved numerically over a range of β. A unique solution exists at the turning point $\beta=\beta_t$ and dual solutions are found in the region $\beta>\beta_t$ where $\beta_t=-0.657$ is the turning point in the parametric shear stress curve separating upper from lower branch solutions. An analysis of solutions near the Agrawal point $\beta=0$ is provided, and the large-β asymptotic behavior of solutions is determined. Sample velocity profiles along both solution branches are presented. A linear temporal stability analysis reveals that solutions along the upper branch are stable while those on the lower branch are unstable.

1. Introduction

The boundary-layer flow over a continuously stretching surface is an often-encountered problem in engineering processes. There are many applications in industries such as hot rolling, wire drawing, and glass-fiber production to name but a few. For these and other applications of continuously stretching surfaces, see Refs. [1–3].

Axisymmetric stagnation point flow on a flat surface was first studied by Homann [4]. Many variations on that problem have been studied including the effects of wall stretching and suction and blowing through a porous wall; see the review by Wang [5]. Of interest here is the paper by Mahapatra and Gupta [6] who studied Homann stagnation-point flow on a radially stretching surface, a procedure used for cooling a stretching surface. Wang [7] subsequently considered, *inter alia*, Homann stagnation-point flow on a radially shrinking surface.

A new axisymmetric stagnation-point flow, also an exact solution of the Navier–Stokes equations, was discovered by Agrawal [8]. In contrast to the irrotational outer flow of Homann [4], this flow is rotational in the far field. Agrawal [8] derived his solution using spherical coordinates. The nature of the flow becomes most apparent when cylindrical coordinates (r, z) with coordinate velocities (u, w) are used which furnish the pleasingly simple solution

$$u(r,z)=2a\,r\,z, \quad w(r,z)=-2a\,z^2 \tag{1.1}$$

in which the parameter a having units $(LT)^{-1}$ measures the strength of the stagnation-point flow. It is clear that the impermeable and no-slip conditions are satisfied at the wall $z=0$.

This work continues a series of studies on applications of Agrawal [8] stagnation-point flows in different situations. In the first study, Weidman [9] analyzed Agrawal stagnation-point flow impinging normal to a rotating plate, thus extending the work of Hannah [10] who was the first to analyze Homann [4] stagnation-point flow impinging on a rotating plate. In a second study Weidman [11] considered Agrawal stagnation-point flow impinging on a quiescent liquid surface, thus extending the work of Wang [12] who did the same problem using the classic Homann stagnation-point flow. In the present study, Agrawal stagnation-point flow impinging normal to a radially stretching sheet is investigated. Solutions to this problem depend on a single parameter β which measures the radial stretching $(\beta>0)$ or shrinking $(\beta<0)$ rate of the plate. Dual solutions are found and these solutions are tested for linear temporal stability.

The presentation is as follows. The problem is formulated in Section 2. Analysis near the Agrawal solution and the large β asymptotics are given in Section 3. The results of numerical calculations of the governing equation presented in Section 4 reveal dual solutions so the linear temporal stability of those solutions is computed. The paper ends with a discussion and concluding remarks in Section 5.

2. Problem formulation

The problem is formulated using cylindrical coordinates (r,θ,z) with corresponding velocities (u,v,w). The stretching surface is located at $z=0$ and the domain of flow is in the upper half plane. The flow is axisymmetric $(\partial/\partial\theta=0)$ and has no swirl $(v=0)$. A

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reduction of the governing equations that gives rise to axisymmetric rotational Agrawal [8] stagnation-point in the far field above the surface, and satisfies the equation of continuity for incompressible flow, has the form [9,11]

$$u(r,\eta)=a^{2/3}\nu^{1/3}rF'(\eta),\quad w(\eta)=-2a^{1/3}\nu^{2/3}F(\eta),\quad \eta=\left(\frac{a}{\nu}\right)^{1/3}z \tag{2.1}$$

where a prime denotes differentiation with respect to η and ν is the kinematic viscosity of the fluid. Inserting this solution *ansatz* into
the Navier–Stokes equation furnishes the boundary-value problem for $\eta\geq 0$ as

$$F'''+2FF''-F'^2=0,\quad F(0)=0,\quad F'(0)=\beta,\quad F''(\infty)=2 \tag{2.2}$$

where β is a non-dimensional parameter measuring radial stretch rate of the horizontal sheet.

The pressure field found by integrating the z-component of the Navier–Stokes equation is

$$p(\eta)=p_0-2a^{2/3}\rho\nu^{4/3}\left(F^2(\eta)+F'(\eta)-\beta\right) \tag{2.3}$$

in which p_0 is the horizontally uniform stagnation pressure.

The radial wall shear stress, here denoted as τ_r, is given by

$$\tau_r=\rho\,\nu\frac{\partial u}{\partial z}\bigg|_{z=0}=\rho\,\nu\,a\,r\,F''(0). \tag{2.4}$$

3. Analysis near the Agrawal solution and large β asymptotics

In this section the parametric solution behavior near $\beta=0$ corresponding to the exact Agrawal solution $f(\eta)=\eta^2$ is analyzed and asymptotic results are provided for large values of β.

3.1. Analysis in the neighborhood of the Agrawal solution

The nature of the parametric solution curve near $\beta=0$ may be analyzed by introducing the perturbation expansion developed in powers of $\beta\ll 1$

$$F(\eta)=\left(\eta+\frac{\beta}{2}\right)^2-\beta^2 f_2(\eta) \tag{3.1}$$

and its derivatives into Eq. (2.2). This yields, at $O(\beta^2)$, the problem

$$f_2'''+2\eta^2 f_2''-4\eta f_2'+4f_2=0 \tag{3.2a}$$

$$f_2(0)=\tfrac{1}{4}, \tag{3.2b}$$

$$f_2'(0)=0, \tag{3.2c}$$

$$f_2''(\infty)=0. \tag{3.2d}$$

By the simple expedient of differentiation, one finds the separable equation

$$f_2^{iv}+2\eta^2 f_2'''=0 \tag{3.3}$$

two integrations of which yields

$$f_2''=A\int_\eta^\infty e^{-\frac{2}{3}t^3}\,dt \tag{3.4}$$

satisfying $f_2''(\infty)=0$ and A is an integration constant. Since we need $f_2''(\eta)$ only near $\eta=0$, write

$$f_2''(\eta)=A\left[I-\int_0^\eta e^{-\frac{2}{3}t^3}\,dt\right] \tag{3.5}$$

where the definite integral is given in terms of the Gamma function $\Gamma(z)$ defined in Abramowitz and Stegun [13], namely

$$I=\int_0^\infty e^{-\frac{2}{3}t^3}\,dt=\left(\frac{3}{2}\right)^{1/3}\Gamma\left[\frac{4}{3}\right]. \tag{3.6}$$

Now we expand the integrand of the definite integral and integrate twice to obtain

$$f_2(\eta)=C+B\eta+A\left[I\frac{\eta^2}{2}-\frac{\eta^3}{6}+O(\eta^6)\right]. \tag{3.7}$$

Boundary conditions (3.2b, c) require $B=0$ and $C=1/4$.

The last arbitrary constant is determined by evaluating (3.2a) at $\eta=0$ in order to recover the constant sacrificed in the above differentiation; this gives $A=1$. Evaluation of (3.5) at $\eta=0$, and use of the relation $F''(0)=1-\beta^2 f_2''(0)$, yields the desired result

$$F''(0)=2-I\beta^2 \tag{3.8}$$

confirming the downward concavity of the numerically determined parametric curve $F''(0)$ at the Agrawal solution point $\beta=0$.

3.2. Large β asymptotics

The goal in this section is to find a two-term large-β asymptotic formula for the wall shear stress. This is accomplished by matching inner and outer solutions.

Inspection of Eq. (2.2), with $\beta\gg 1$, suggests the rescaling

$$\eta=\beta^{-\alpha}\xi,\quad F(\eta)=\beta^\alpha f(\xi) \tag{3.9}$$

which maintains the form of the differential equation in (2.2) and yields

$$f'''+2ff''-f'^2=0,\quad f(0)=0,\quad f'(0)=\beta^{1-2\alpha},\quad f''(\infty)=2\beta^{-3\alpha} \tag{3.10}$$

where a prime now denotes differentiation with respect to ξ. We set $\alpha=1/2$ to obtain velocities of order unity, *viz.* $f'(0)=1$. This suggests the first term in an expansion for $F(\eta)$ given by

$$F=\beta^{1/2}f_0(\xi)+\cdots \tag{3.11}$$

To ascertain the next term in the expansion we seek and outer solution of (2.2) for $\eta=O(1)$, to find that

$$F(\eta)=(\eta+C(\beta))^2 \tag{3.12}$$

where C is a constant to be determined. Matching to the leading-order inner solution (3.11) leads to

$$C(\beta)=f_{0\infty}^{1/2}\beta^{1/4} \tag{3.13}$$

where $f_0\to f_{0\infty}$ as $\xi\to\infty$. Consequently, as $\eta\to 0$ we have the development

$$F=f_{0\infty}\beta^{1/2}+2f_{0\infty}^{1/2}\beta^{-1/4}+\cdots \tag{3.14}$$

which therefore requires that the expansion for $F(\eta)$ be of the form

$$F=\beta^{1/2}f_0(\xi)+\beta^{-1/4}f_1(\xi)+\cdots \tag{3.15}$$

Inserting (3.15) into boundary-value problem (2.2) gives

$$f_0'''+2f_0f_0''-f_0'^2=0,\quad f_0(0)=0,\quad f_0'(0)=1,\quad f_0''(\infty)=0 \tag{3.16a}$$

$$f_1'''+2(f_0f_1''-f_0'f_1'+f_0''f_1)=0,\quad f_1(0)=0,\quad f_1'(0)=0,\quad f_1''(\infty)=2f_{0\infty}^{1/2}. \tag{3.16b}$$

Then the leading behavior of the wall shear stress is given as

$$F''(0)\sim f_0''(0)\beta^{3/2}+f_1''(0)\beta^{3/4}+\cdots \tag{3.17}$$

A standard shooting method was implemented to solve these boundary-value problems using the ODEINT integration and MNEWT Newton iteration codes provided in Press et al. [14]. Guesses on the wall shear stress parameters are made until the far-field condition in Eq. (3.12) is satisfied. Integrations were carried out changing the length of integration η_{max} to ensure solution