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Group classification of one-dimensional equations of capillary fluids where the specific energy is a function of density, density gradient and entropy

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ABSTRACT

An application of group analysis provides a regular procedure for mathematical modeling by classifying differential equations with respect to arbitrary elements. This paper presents the group classification of one-dimensional equations of fluids where the internal energy is a function of the density, the gradient of the density and the entropy. The group classification separates all models into 83 different classes according to the admitted Lie group. Some invariant solutions are studied.

1. Introduction

Gradient models have been widely discussed in continuum mechanics during the last decades. These models are derived by including the gradient of internal parameters into the set of independent constitutive variables. The gradient theories have been developed in mechanics for treating various phenomena, such as capillarity in fluids, liquids in nanotubes, plasticity and friction in granular materials or shear band deformations, poromechanics.

One of the approaches for deriving the governing equations of models in continuum mechanics is the variational approach which uses Hamilton's principle of stationary action. The advantage of the variational method is that the knowledge of only one scalar function written in terms of the constitutive variables, the Lagrangian of the system, allows one to obtain a closed system of governing equations. However, it is in general not an easy task to find an explicit form of the Lagrangian in terms of the constitutive variables. To do so, a number of hypotheses should be formulated. The hypotheses are developed from the analysis of experimental data and the mathematical analysis of the studied models, in particular their symmetries.

Symmetries have always attracted the attention of scientists. It is noteworthy that group properties play a fundamental role in modern micro and macro physics, whereas there is lack of studies in continuum mechanics which employ symmetries. One of the tools for the study of symmetries is the group analysis method [1–3], which is a basic method for constructing exact solutions of partial differential equations. A wide range of applications of group analysis to partial differential equations has been collected in [4–6].

For mathematical physics and continuum mechanics equations it is characteristic to possess classical symmetries, for example, the Galilean group of transformations. Symmetries of a particular continuum mechanics model can usually be extended, and the more extensions are available, the more possible invariant solutions can be obtained.

An extension of symmetries can be obtained by specifying constitutive relations. Group analysis, besides facilitating the construction of exact solutions, provides a regular procedure for mathematical modeling by classifying differential equations with respect to arbitrary elements related with constitutive equations. The presence of symmetry properties allows one to specify these elements. It is worth to notice here that the experimentally determined functions are not strictly fixed either: there are some arbitrary elements in their definition which can also be specified with the requirement to possess a particular class of symmetries. Requirements of having symmetries are one of the main tools for modeling by the group analysis method. For example, except for

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a Galilean group of transformations, group analysis makes it possible to impose (as the simplicity criterion) the requirement that an arbitrary element be such that, with it, the differential equations of a model admit a group with specific properties. This feature of group analysis is the fundamental basis for the mathematical modeling in the present paper.

The present manuscript is focused on the group classification of the governing equations whose specific internal energy ε is a function of the density, the density gradient, and the entropy [7],[1]

$$\rho_t+(\rho u)_x=0,\quad (\rho u)_t+(\rho u^2+\Pi)_x=0,\quad (\rho s)_t+(\rho s u)_x=0,$$
$$\Pi=\rho\frac{\delta(\rho\varepsilon)}{\delta\rho}-\rho\varepsilon=\rho^2\varepsilon_\rho-2\rho(\rho\rho_x\varepsilon_\alpha)_x+2\rho\rho_x^2\varepsilon_\alpha,\quad \alpha=\rho_x^2, \tag{1}$$

where t is the time, ∇ is the gradient operator with respect to the space variables, ρ is the fluid density, u is the velocity field, s is the entropy, $\varepsilon(\rho,\alpha,s)$ is a given internal energy, and $\delta/\delta\rho$ denotes the variational derivative with respect to ρ at a fixed value of u. These models were studied in [8–13]. A review of these models can be found in [7,14] and references therein. In the present paper we follow [7], where Eqs. (1) were obtained using the Lagrangian of the form

$$L=\tfrac{1}{2}|\mathbf{u}|^2-\varepsilon(\rho,\nabla\rho,s).$$

These models are examples of a continuum where the behavior depends not only on the thermodynamical variables but also on their derivatives with respect to space and time. Another set of models where the behavior of the medium depends not only on the thermodynamical variables but also on their derivatives with respect to space and time was constructed in [15,16], using the Lagrangian of the form

$$L=\tfrac{1}{2}|\mathbf{u}|^2-W(\rho,\dot\rho,s)$$

where $\dot{}=\partial/\partial t+\mathbf{u}\nabla$. If there is no dependence on the derivatives, then these models correspond to the classical gas dynamics model ($\varepsilon=\varepsilon(\rho,s)$). An exhaustive program of studying the models appearing in the group classification of the gas dynamics equations was announced in [17]. It is known that for arbitrary function $\varepsilon(\rho,s)$ the gas dynamics equations admit a Galilean group extended by the group of homothetic transformations. According to those extensions, the function $\varepsilon(\rho,s)$ is separated out into 13 different classes. It is worth to notice that one of these classes is the class of polytropic gases, which is widely used in applications. Some results of the program [17] were summarized in [18].

An application of group analysis employs several steps. The first step is group classification with respect to arbitrary elements. An algorithm of the group classification is applied in cases where a system of differential equations has arbitrary elements in the form of undefined parameters and functions. This step is necessary since a specialization of the arbitrary elements can lead to different admitted Lie groups. In particular, group classification selects the functions $\varepsilon(\rho,|\nabla\rho|,s)$ such that the fluid dynamics equations (1) possess additional symmetry properties extending the kernel of the admitted Lie groups. Algorithms of finding equivalence and admitted Lie groups are particular parts of the algorithm of the group classification.

A complete group classification of Eqs. (1), where $\varepsilon=\varepsilon(\rho,|\nabla\rho|)$, is performed in [19]. Invariant solutions of some particular cases are considered there. Group classification of the class of models describing the behavior of a dispersive continuum with $W=W(\rho,\dot\rho)$ was studied in [20] (one-dimensional case) and [21] (three-dimensional case). Invariant solutions of some particular cases which are separated out by the group classification are considered in [20,22]. The group classification performed in these studies [20,19,21] follows the classical method developed by Ovsiannikov [3] for the group classification of the gas dynamics equations.

The classical approach [3] to non-isentropic equations of fluids with internal inertia is very complicated. Even the study of particular cases leads to cumbersome investigations [23]. In the present paper we use an algebraic approach for the group classification of non-isentropic equations of fluids with internal inertia.

The algebraic approach takes the algebraic properties of an admitted Lie group and the knowledge of the algebraic structure of admitted Lie algebras into account, and allows for significant simplification of the group classification. In particular, the group classification of a single second-order ordinary differential equation, done by the founder of the group analysis method, Lie [24,25], cannot be performed without using the algebraic structure of admitted Lie groups. Recently, the algebraic properties have been applied for group classification [26–33].

The present paper is focused on the group classification of the one-dimensional equations of fluids (1), where $\varepsilon=\varepsilon(\rho,|\nabla\rho|,s)$ with $\varepsilon_s\neq 0$.

This paper is organized as follows. The equivalence Lie group of transformations is presented in Section 2. The equivalence transformations are applied for simplifying the function $\varepsilon(\rho,|\nabla\rho|,s)$ in the process of the classification. We classify all models with respect to the admitted Lie groups in Section 3, where we consider 2 cases. In the first case, where $k_4\neq 0$, the analysis is similar to the group classification of the gas dynamics equations. In the second case, where $k_4=0$, the analysis uses the idea of the algebraic approach which separates the study of group classification into two steps. In the first step, one makes a preliminary study of possible coefficients of the basis generators using the requirement of admitted generators to compose a Lie algebra. In the second step, one substitutes these coefficients of each generator of the Lie algebra into the determining equation. Solving the system of equations obtained, the function $\varepsilon(\rho,|\nabla\rho|,s)$ and additional restrictions for the coefficients of the basis generators are obtained. The results of the group classification and the admitted Lie algebras of Eq. (1) are summarized in Table 1.

2. Equivalence Lie group

An equivalence Lie group allows changing arbitrary elements conserving the structure of the studied equations. An infinitesimal operator X^e of the equivalence Lie group is sought in the form [34]

$$X^e=\xi^x\partial_x+\xi^t\partial_t+\zeta^\rho\partial_\rho+\zeta^u\partial_u+\zeta^\alpha\partial_\alpha+\zeta^s\partial_s+\zeta^\varepsilon\partial_\varepsilon,$$

where the coefficients ξ^x, ξ^t, ζ^ρ, ζ^u, ζ^α, ζ^s and ζ^ε are all functions of $(x,t,\rho,u,\alpha,s,\varepsilon)$.

Calculations give the following basis of generators of the equivalence Lie group:

$$X_1^e=\rho\partial_\rho+2\alpha\partial_\alpha,\quad X_2^e=t\partial_t+x\partial_x-2\alpha\partial_\alpha,$$
$$X_3^e=t\partial_x-u\partial_u-2\varepsilon\partial_\varepsilon,\quad X_4^e=f(s)\partial_s,$$
$$Y_1^e=\partial_t,\quad Y_2^e=\partial_x,\quad Y_3^e=t\partial_x+\partial_u,\quad Z_1^e=\rho^{-1}\partial_\varepsilon,$$
$$Z_2^e=f(s)\partial_\varepsilon,\quad Z_3^e=g(\rho)\sqrt{\alpha}\partial_\varepsilon,$$

where the functions $f(s)$ and $g(\rho)$ are arbitrary.

Since the equivalence transformations corresponding to the operators X_3^e, X_4^e, Z_1^e, Z_2^e and Z_3^e are applied for simplifying the function ε in the classification process, let us present these transformations. Because the function ε depends on ρ, α and s, only the transformations of these variables are presented:

$$X_3^e:\tilde\rho=\rho,\quad \tilde\alpha=\alpha,\quad \tilde s=s,\quad \tilde\varepsilon=e^{-2a}\varepsilon,$$
$$X_4^e:\tilde\rho=\rho,\quad \tilde\alpha=\alpha,\quad \tilde s=h(s,a),\quad \tilde\varepsilon=\varepsilon,$$

[1] See also references therein.