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An equivalence between generalized Maxwell model and fractional Zener model

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ABSTRACT

Both classic rheological models and fractional derivative models have been widely adopted to model the viscoelastic behaviors of materials. In this work, we present a detailed comparison of the performance of the generalized Maxwell model and fractional Zener model. We first describe a method to determine the parameters of the generalized Maxwell model from the fractional Zener model based on the equivalence of complex modulus in the frequency domain of the two models. The two models are then applied to investigating the stress response under constant strain rate, stress relaxation, cyclic and random loading conditions. The simulation results of the two models show excellent quantitatively equivalence. This finding can provide insight into choosing the more suitable model for specific conditions.

1. Introduction

Viscoelasticity represents that the behavior of materials is intermediate between linear solids and viscous liquid (Ferry, 1980). When deformed, most of polymers and biological tissues exhibit this time-dependent viscous behavior represented as a stiffer stress response at a higher loading rate and a more compliant response at a lower loading rate. To understand the origin of viscoelasticity, various physical-based models have been proposed (Rouse et al., 1953; De Gennes, 1979; Doi and Edwards, 1988; Li et al., 2015). For example, the Rouse model (Rouse et al., 1953) was used to explain the properties of unentangled polymer solutions and melts. The reptation model (De Gennes, 1979; Doi and Edwards, 1988) was used to explain the relaxation and viscosity of entangled polymeric materials. However, for engineering applications, the most widely used method to describe the viscoelasticity is based on rheological models.

The viscoelastic rheological models contain the elastic components modeled as springs and the viscous components modeled as dashpots (Ferry, 1980). Based on the arrangement of these components, various models have been developed, such as the Maxwell model, the Kelvin-Vogit model, the Zener model and more complex generalized Maxwell model. The generalized Maxwell model contains an elastic spring in parallel with multiple Maxwell models to represent the relaxation occurring at a broad distribution of time. This model has been successfully applied to studying various viscoelastic solids (Del Nobile et al., 2007; Kaufman et al., 2008; Yu et al., 2014; Xiao et al., 2015). For example, Del Nobile et al. (2007) has used the generalized Maxwell model to fit the experimental data of five different food matrices. Kaufman et al. (2008) has applied the generalized Maxwell model to studying the stress relaxation of hydrogels, while Yu et al. (2014) and Xiao et al. (2015) have employed this model to describe the shape-memory behaviors of amorphous polymers.

Though the classic viscoelastic models can well describe the experimental results, they typically involve excessive number of parameters. It is shown that the viscoelastic models can be generalized into fractional derivative models (Koeller, 1984; Bagley and Torvik, 1986; Schiessel et al., 1995). In recent years, fractional models have been widely adopted in the field of diffusion (Wu, 2012; Wang et al., 2010; Zhao and Sun, 2011), heat transfer (Jiang and Qi, 2012), chaos (Baleanu et al., 2015; Wu and Baleanu, 2015) and nonlocal elasticity (Tarasov, 2014). However, the most extensive application of fractional models still lies in the field of linear viscoelasticity (Mainardi, 2010). The general procedure to obtain the fractional viscoelastic model is through replacing the derivative of order 1 of the dashpot with the fractional derivative of order between 0 and 1. Through this process, various fractional derivative models can be obtained, such as fractional Maxwell model, fractional Kelvin–Vogit model, fractional Zener model (Mainardi, 2010) and more complex model as shown in Arikoglu (2014). These models have been widely adopted to describe the relaxation and creep behaviors of elastomers (Di Paola et al., 2011) and natural materials (Cataldo et al., 2015), dynamic behavior of biological tissue (Kohandel et al., 2005) and other solids (Rossikhin and Shitikova, 2010), and visco-elastic Euler–Bernoulli beam (Di Paola et al., 2013). In addition, Fan et al. (2015) and Yu et al. (2015) have

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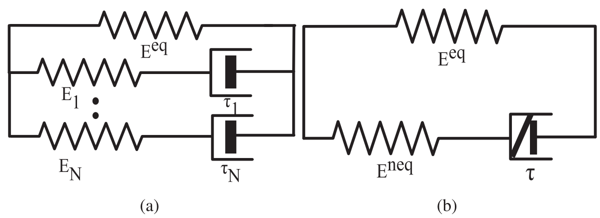

Fig. 1. Rheological representative of (a) generalized Maxwell model and (b) fractional Zener model.

developed numerical algorithm to obtain the model parameters for fractional derivative models. Though both rheological models and fractional models can be applied to describe the viscoelastic behaviors, limited work has been done to compare their performance in detail.

In this work, we present a numerical study to compare the performance of generalized Maxwell model and fractional Zener model. The model descriptions are pesented in Section 2. The following section describes the procedure of obtaining the model parameters of generalized Maxwell model from an approximation between the dynamic modulus of the two models. Finally, we compare the performance of the two models under four different types of loading conditions: constant strain rate, stress relaxation, cyclic loading and a random loading condition.

2. Constitutive model

2.1. Generalized Maxwell model

The rheological representative of generalized Maxwell model is shown in Fig. 1a, which is composed of a spring to describe the equilibrium elastic response and multiple Maxwell elements arranged in parallel to represent the viscoelastic response.

In each Maxwell element, the total strain of the spring ε_j^e and the dashpot ε_j^v should be equal to the strain ε in the elastic branch, which yields,

$$\varepsilon = \varepsilon_j^e + \varepsilon_j^v, \quad j = 1..N, \tag{1}$$

where N is the total number of Maxwell elements.

The total stress is given by,

$$\sigma = E^{eq}\varepsilon + \sum_j^N E_j \varepsilon_j^e, \tag{2}$$

where E^{eq} is the equilibrium elastic modulus and E_j is the modulus of the spring in jth Maxwell element.

The following linear evolution equation is adopted for ε_j^v,

$$\dot{\varepsilon}_j^v = \frac{\varepsilon - \varepsilon_j^v}{\tau_j}, \quad \varepsilon_j^v(t=0) = 0, \tag{3}$$

where τ_j is the relaxation time of the dashpot in jth Maxwell element.

Eqs. (1)–(3) complete the generalized Maxwell model, which contains the following parameters: equilibrium elastic modulus E^{eq} and viscoelastic relaxation spectrum (τ_j, E_j).

2.2. Fractional Zener model

The 1D rheological representative of fractional Zener model is shown in Fig. 1b, which is consisted of an equilibrium elastic spring in parallel with a fractional damping Maxwell element.

Similarly, the total strain in the non-equilibrium branch equals to that of the equilibrium branch, which gives,

$$\varepsilon = \varepsilon^e + \varepsilon^v. \tag{4}$$

The stress response is given by,

$$\sigma = E^{eq}\varepsilon + E^{neq}\varepsilon^e, \tag{5}$$

where E^{eq} is the modulus of the equilibrium elastic spring and E^{neq} is the modulus of the spring in the non-equilibrium fractional damping Maxwell element.

The evolution of ε^v in the fractional damping element can be described as Haupt et al. (2000),

$$\frac{d^\alpha \varepsilon^v}{dt^\alpha} = \frac{\varepsilon - \varepsilon^v}{\tau^\alpha}, \quad \varepsilon^v(t=0) = 0, \tag{6}$$

where $\frac{d^\alpha(\cdot)}{dt^\alpha}$ is the Riemann–Liouville fractional derivative with $0 < \alpha < 1$, which is defined as (Haupt et al., 2000; Nguyen et al., 2010),

$$\frac{d^\alpha(f)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(s)}{(t-s)^\alpha}ds, \quad f(t=0) = 0. \tag{7}$$

$\Gamma(x)$ is the Eulerian Gamma function.

Eqs. (4)–(6) complete the fractional Zener model.

3. Relaxation spectrum of generalized Maxwell model

The procedures of obtaining viscoelastic relaxation spectrum of generalized Maxwell model from fractional derivative model have been discussed in detail in Haupt et al. (2000); Nguyen et al. (2010); Xiao et al. (2013) and Xiao and Nguyen (2015). Here we briefly summarized the main processes.

The dynamic storage and loss modulus of the two models under a small sinusoidal oscillations can be represented as,

$$\begin{aligned}
E'_{gene}(\omega) &= E^{eq} + \sum_j^N \frac{E_j\omega^2\tau_j^2}{1+\omega^2\tau_j^2},\\
E'_{frac}(\omega) &= E^{eq} + \frac{E^{neq}((\omega\tau)^{2\alpha} + (\omega\tau)^\alpha\cos(\alpha\pi/2)}{1+(\omega\tau)^{2\alpha} + (\omega\tau)^\alpha\cos(\alpha\pi/2)},\\
E''_{gene}(\omega) &= \sum_j^N \frac{E_j\omega\tau_j}{1+\omega^2\tau_j^2},\\
E''_{frac}(\omega) &= \frac{E^{neq}(\omega\tau)^\alpha\sin(\alpha\pi/2)}{1+(\omega\tau)^{2\alpha} + (\omega\tau)^\alpha\cos(\alpha\pi/2)},
\end{aligned} \tag{8}$$

where ω is the angular frequency, $E'(\omega)$ denotes the storage modulus, and $E''(\omega)$ denotes the loss modulus.

The viscoelastic spectra $h(\nu)$ can be calculated from the complex moduli $E^*(i\omega) = E'(\omega) + iE''(\omega)$ using the inverse Stieltjes transform (Christensen, 2003),

$$\frac{E^*(i\omega)}{i\omega} = \int_0^\infty \frac{h(\nu)}{\nu + i\omega}d\nu, \tag{9}$$

where ν is the relaxation frequency, which is inverse of the relaxation time.

The cumulative relaxation spectra are defined as $H(\nu) = \int_0^\nu h(u)du$, which yields the cumulative spectra of the two models,

$$\begin{aligned}
H_{gene}(\nu) &= \sum_j^N E_j\langle \nu - \nu_j\rangle,\\
H_{frac}(\nu) &= \frac{E^{neq}}{\alpha\pi}\left[\arctan\left(\frac{(\nu\tau)^\alpha + \cos(\alpha\pi)}{\sin(\alpha\pi)}\right) - \pi\left(\frac{1}{2} - \alpha\right)\right],
\end{aligned} \tag{10}$$

where $\langle \nu - \nu_j\rangle = 1$ for $\nu \geq \nu_j$ and $\langle \nu - \nu_j\rangle = 0$ for $\nu < \nu_j$.

A power law distribution of relaxation frequencies is then assumed,

$$\nu_j^0 = \nu_{min}^0\left(\frac{\nu_{max}^0}{\nu_{min}^0}\right)^{\frac{j-1}{N-1}}. \tag{11}$$