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Constrained mechanical systems and gradient systems with strong Lyapunov functions

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ABSTRACT

The characteristics of stationary and non-stationary gradient systems with strong Lyapunov functions are studied. The conditions under which holonomic mechanical systems, generalized Birkhoff systems and generalized Hamilton systems can be considered as gradient systems are given. The characteristics of the gradient systems can be used to study the stability of the mechanical systems. Some examples are given to illustrate the application of the results.

1. Introduction

Dynamics of constrained mechanical systems has been an important branch of mechanics. In 1894, H.R. Hertz firstly divided the constraints into two types, namely holonomic and nonholonomic constraints [1], and established a new field of mechanics of nonholonomic constrained systems. With the development of science and technology, studies on constrained mechanical systems have become a popular subject in classical mechanics [2,3], field theory[4], relativistic mechanics [5–7], spacecraft attitude dynamics [8], robots control theory [9], machinery engineering [10], symmetry, conserved quantities and symmetrical perturbation [11–13] as well as invariant manifolds [14], etc.

Gradient systems are especially suitable for being studied by using Lyapunov functions [15]. R.I. McLachlan and his co-workers put forward the gradient systems with strong Lyapunov functions [16], which are more general gradient systems than the gradient systems in [15]. They are useful in the study of the mechanical systems using gradient systems. Recently, the study of the gradient system includes a series of contributions such as the general gradient system representation and the skew-gradient system representation for some mechanical systems [17–22], the gradient system with a strict Lyapunov function [23] as well as a generalization of a gradient system [24].

In this paper gradient systems with strong Lyapunov functions and their characteristics are discussed firstly, and they are generalized to nonautonomous systems. Then the holonomic mechanical system, the generalized Birkhoff system and the generalized Hamilton system are transformed into gradient systems, so that one can use the characteristics of gradient systems to study the stability of these mechanical systems.

2. Gradient systems with strong Lyapunov functions

The differential equations of the gradient systems with strong Lyapunov functions are [16]

$$\dot{x}_i = S_{ij}\frac{\partial V}{\partial x_j} \quad (i, j = 1, 2, \cdots, m) \tag{1}$$

where $S_{ij} = S_{ij}(\boldsymbol{x})$, the matrix corresponding to S_{ij} is symmetric negative definite and $V = V(\boldsymbol{x})$. If the Eq. (1) are generalized to the case that V includes time, we have

$$\dot{x}_i = S_{ij}(\boldsymbol{x})\frac{\partial V(t, \boldsymbol{x})}{\partial x_j} \quad (i, j = 1, 2, \cdots, m) \tag{2}$$

Eqs. (1) and (2) are called a gradient system I and a gradient system II, respectively.

The gradient system I has the following characteristics:

Proposition 1. If

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$$(S_{ij}) = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix}$$

then the gradient system I becomes a general gradient system.

Proposition 2. For the gradient system I, we have

$$\dot{V} = \frac{\partial V}{\partial x_i} S_{ij} \frac{\partial V}{\partial x_j} \leq 0 \tag{3}$$

if and only if $\frac{\partial V}{\partial x_i} = 0$, then $\dot{V}$ equals zero [16]. So if V is a Lyapunov function then the solutions are asymptotically stable.

Proposition 3. For the gradient system II, we have

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i} S_{ij} \frac{\partial V}{\partial x_j} \tag{4}$$

If V is a Lyapunov function, the sign of $\dot{V}$ can be determined by (4). If $\dot{V}$ is negative definite then the solutions are asymptotically stable.

The characteristics of the two kinds of gradient systems can be used to study the stability of the mechanical systems which can be transformed into the two kinds of gradient systems.

3. Gradient representations for holonomic mechanical systems

3.1. The differential equations of motion

The differential equations of the bilateral ideal holonomic mechanical system are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s \quad (s = 1, 2, \cdots, n) \tag{5}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian of the system, and $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are generalized nonpotentials forces. Assuming the system is nonsingular, and introducing the generalized momenta p_s and Hamiltonian H as follows

$$p_s = \frac{\partial L}{\partial \dot{q}_s}, \quad H = p_s \dot{q}_s - L \quad (s = 1, 2, \cdots, n) \tag{6}$$

the Eq. (5) can be written as

$$\dot{a}^{\mu} = \omega^{\mu\nu} \frac{\partial H(t, \boldsymbol{a})}{\partial a^{\nu}} + \Lambda_{\mu}(t, \boldsymbol{a}) \quad (\mu, \nu = 1, 2, \cdots, 2n) \tag{7}$$

Where

$$a^s = q_s \quad a^{n+s} = p_s$$

$$\left(\omega^{\mu\nu}\right) = \begin{pmatrix} 0_{nxn} & 1_{nxn} \\ -1_{nxn} & 0_{nxn} \end{pmatrix}, \Lambda_s = 0$$

$$\Lambda_{n+s} = \tilde{Q}_s(t, \boldsymbol{a}) \tag{8}$$

here $\tilde{Q}_s$ are represented by canonical variables corresponding to Q_s.

If $\partial H/\partial t = \partial \Lambda_\mu/\partial t = 0$, then Eq. (7) can be written as

$$\dot{a}^{\mu} = \omega^{\mu\nu} \frac{\partial H(\boldsymbol{a})}{\partial a^{\nu}} + \Lambda_{\mu}(\boldsymbol{a}) \quad (\mu, \nu = 1, 2, \cdots, 2n) \tag{9}$$

3.2. Gradient representations for the system

For Eq. (9) if there exists a symmetric negative definite matrix $(S_{\mu\nu}(\boldsymbol{a}))$ and $V = V(\boldsymbol{a})$ satisfying

$$\omega^{\mu\nu} \frac{\partial H(\boldsymbol{a})}{\partial a^{\nu}} + \Lambda_{\mu}(\boldsymbol{a}) = S_{\mu\nu}(\boldsymbol{a}) \frac{\partial V(\boldsymbol{a})}{\partial a^{\nu}} \quad (\mu, \nu = 1, 2, \cdots, 2n) \tag{10}$$

then the system can be considered as the gradient system I.

For Eq. (7) if there exists a symmetric negative definite matrix $(S_{\mu\nu}(\boldsymbol{a}))andV = V(t, \boldsymbol{a})$ satisfying

$$\omega^{\mu\nu} \frac{\partial H(t, \boldsymbol{a})}{\partial a^{\nu}} + \Lambda_{\mu}(t, \boldsymbol{a}) = S_{\mu\nu}(\boldsymbol{a}) \frac{\partial V(t, \boldsymbol{a})}{\partial a^{\nu}} \quad (\mu, \nu = 1, 2, \cdots, 2n) \tag{11}$$

then the system can be considered as the gradient system II.

It is noted that, if Eqs. (10) and (11) cannot be satisfied, then one cannot verify whether the system is a gradient system or not, because it is related to the choice of the first order form of the equations.

3.3. Illustrative example

Example 1. The Lagrangian and the generalized forces of single degree of freedom system are

$$L = \frac{1}{2}\dot{q}^2 - 3q^2(2 + \sin q)^2$$
$$Q = -6\dot{q}(2 + \sin q) + \frac{\dot{q}^2 \cos q}{2 + \sin q} \tag{12}$$

Let's try to transform the system into the gradient system I and study the stability of the zero solution.

The differential equations are

$$\ddot{q} = -6q(2 + \sin q)^2 - 6\dot{q}(2 + \sin q) + \frac{\dot{q}^2 \cos q}{2 + \sin q}$$

Let

$$a^1 = q, \quad a^2 = \frac{\dot{q}}{2 + \sin q} + 2q$$

then

$$\dot{a}^1 = -\left(2a^1 - a^2\right)\left(2 + \sin a^1\right)$$
$$\dot{a}^2 = -2\left(2a^2 - a^1\right)\left(2 + \sin a^1\right)$$

they can be written as

$$\begin{pmatrix} \dot{a}^1 \\ \dot{a}^2 \end{pmatrix} = \begin{pmatrix} -\left(2 + \sin a^1\right) & 0 \\ 0 & -2\left(2 + \sin a^1\right) \end{pmatrix} \begin{pmatrix} \frac{\partial V}{\partial a^1} \\ \frac{\partial V}{\partial a^2} \end{pmatrix}$$

where the matrix is symmetric negative definite and the function V is

$$V = \left(a^1\right)^2 + \left(a^2\right)^2 - a^1 a^2$$

It is positive definite in the neighborhood of $a^1 = a^2 = 0$, so the zero solution $a^1 = a^2 = 0$ is asymptotically stable.