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Surface/interfacial anti-plane waves in solids with surface energy

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ABSTRACT

In this paper we discuss new type of surface anti-plane waves localized near the surface an elastic half-space and in the vicinity of plane interface between two half-spaces, when considering surface strain and kinetic energies. We also consider the case of non-perfect interface, i.e. when a jump of displacement or of its gradient, with the aim of modelling lacking of adhesion between solids. The phase velocity profiles and dispersion relations of surface waves are presented and several different material parameters are considered. Among the results, we observe an anomalous dispersion when the surface/interface is stiffer than the bulk material. These results can be exploited for the nondestructive characterization and the analysis of thin inter-phases between two solids, and can find several engineering applications.

1. Introduction

Surface and interfacial waves, *i.e.* waves whose amplitude decays exponentially with distance from the surface or interface, are well known for being an important subject of study in mechanics [1–3]. The main reason is that as waves propagate, they carry information about geometry and mechanical properties of the media. Their peculiar features are indeed studied and exploited in different fields, mainly seismology, signal processing or nondestructive evaluation.

The most well known example of surface wave is the Rayleigh wave, which propagates near the free interface of an elastic solid. This wave is not dispersive, *i.e.* its phase velocity does not depend on the frequency. More complex is the case of Stoneley waves, which may exist near the interface separating two elastic halfspaces, or they always exist at the interface between a solid and a fluid. In the latter case, when phase velocities of the two media fulfil specific condition, a Leaky-Rayleigh wave can exist. The specific feature of a Leaky-Rayleigh wave is that it is attenuated in the direction of propagation, *i.e.* along the surface/interface, due to the fact that some amount of energy is leaking into the fluid. All these surface waves have something in common: the displacement vector field always belongs to a plane perpendicular to the surface, namely the sagittal plane. Using a well known formalism, we can say that only Pressure (P-) waves and Shear Vertical (SV-) waves are involved. In fact, this kind of surface waves can be seen as the result of a particular linear combination of two aforementioned solutions of the elastodynamic problem when a wavenumber matching occurs. In all cases of free halfspace, solid–solid or solid–fluid interfaces, no surface waves are associated to what are called anti-plane (AP-), also known as Shear Horizontal (SH-) waves, which are polarized perpendicularly with respect to the sagittal plane.

Following the literature, in order to observe such surface waves, we need to extend the analysis to layered media, and consider for instance an additional layer covering the free surface of the solid, or separating the two solids. These results are historically related to seismic waves and were initially developed for characterizing the layer structure of the Earth [4]. In this case we can observe the so called Love waves.

In this work we will consider solids separated by a thin interfacial layer represented by a 2D surface with associated elastic properties. This configuration is of great interest in several engineering applications, and it is particularly suited for modelling solids separated by a softer thin elastic layer. We study antiplane waves localized near half-space free surface or near plane interface between solids. The classic analysis of antiplane waves can be found for example in [1], waves in solids with surface stresses are analyzed in [5–8]. The crucial point of the surface/interface elasticity is the formulation of constitutive equation for surface/interface

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strain energy. Direct approach to the formulation of such constitutive equation is given by Gurtin and Murdoch [9], Steigmann and Ogden [10]. Asymptotic derivation of the interface properties using thinness of the interface is given for example in [11] where other references can be found. Let us note that as in the theory of shells, asymptotic approach requires certain assumptions such as homogeneity or isotropy of thin interface layer as in [11]. More complex microstructured interface models were considered in [12,13] with focusing on the waves propagation. Such microstructured interfaces may be interesting for example for design of acoustic metamaterials, see also recent review [14]. For structured interfaces more complex compatibility conditions across the interface appear related to stiff and soft models of interface. Influence of initial stresses in interface on the wave propagation is analyzed in [15,16]. Recently, analysis of antiplane waves in piezoelectric materials is given in [17,18]. Within the framework of the second gradient elasticity SH surface waves are studied in [19,20].

One possible application of the present study is the characterization of the properties of the interphase between bone and an implant right after surgery. Indeed, this interphase is constituted by a thin newly formed bone, which is softer that mature bone and whose mechanical properties play an essential role in the stability of the implant, and as consequence on the success of the surgery [21,22]. Since the model of surface elasticity by Gurtin and Murdoch [9] found numerous applications in micro- and nanomechanics [23–25,14] our model can be also used for modelling of wave propagation in structures of micron and nanometer size.

The paper is organized as follows. After this introduction (Section 1), in Section 2 we present the governing equations of the model of an isotropic solid with surface/interfacial reinforcements. For the material behaviour in the bulk, we use the classic Hooke's law while for surface/interface we propose the model of surface elasticity. The model is similar to models of Gurtin–Murdoch and Steigmann–Ogden but includes also surface mass and new terms in the case of interfaces between solids. Here, we consider two cases called perfect interface and non-perfect one. For a perfect interface, we assume that displacement field is a continuous in the vicinity of the interface while for a non-perfect interface, we assume that discontinuities in displacements may exist. The motion equations and dynamic boundary/interfacial conditions at the surface/interface are obtained using the variational principle of least action. For an elastic half-space, in Section 3, we derive the solution of the problem which decays exponentially with distance from the half-space surface and discuss range of material parameters when such type solutions exist. The discussed waves are similar to the Love waves existing in layered half-space. Finally, in Section 4 we discuss the propagation surface anti-plane waves along interface between two elastic half-spaces, when considering perfect and non perfect boundary conditions. Finally, in Section 5, some conclusions are drown.

2. Governing equations

Let an elastic solid occupy a volume V in R^3 with the boundary $A = \partial V$ and where R^3 designates the three-dimensional space. We assume that the solid may consist of two parts, V^+ and V^- separated by a smooth interface I. We attribute to the interface surface, strain and kinetic energy densities. In addition, we also assume the presence of surface energy and surface stresses on the part of boundary, that is on $A_s \subset A$. We consider infinitesimal deformations of the solid described by the displacement field

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t), \tag{1}$$

where $\mathbf{u}$ is a twice differentiable vector-function of displacements, $\mathbf{x}$ is the position vector and t is time.

In what follows, we use the classic constitutive equations of an isotropic body in the bulk

$$\begin{aligned} \mathcal{W} &= \mu \mathbf{e} : \mathbf{e} + \frac{1}{2}\lambda(\operatorname{tr}\mathbf{e})^2, \\ \boldsymbol{\sigma} &\equiv \frac{\partial \mathcal{W}}{\partial \mathbf{e}} = 2\mu\mathbf{e} + \lambda\,\mathbf{I}\operatorname{tr}\mathbf{e}, \\ \mathbf{e} &= \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T), \end{aligned} \tag{2}$$

where $\mathcal{W}$ is the strain energy density, $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{e}$ is the strain tensor, $\mathbf{I}$ is the unit second-order tensor, the double dot stands for scalar (inner) product of two second-order tensors, λ and μ are Lamé moduli, $\mu > 0$, $3\lambda + 2\mu > 0$, ∇ is the 3D nabla operator, and tr is the trace operator. The kinetic energy density is given by

$$\mathcal{K} = \frac{1}{2}\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}, \tag{3}$$

where ρ is the mass volume density and overdot stands for the derivative with respect to time t.

For the surface elasticity model, we consider the Gurtin–Murdoch approach [5,9] with taking into account the surface-related kinetic energy. According [9], the surface strain energy density $\mathcal{W}_s$ and surface stress tensor $\boldsymbol{\tau}$ are defined as follows

$$\begin{aligned} \mathcal{W}_s &= \mu_s\boldsymbol{\epsilon} : \boldsymbol{\epsilon} + \frac{1}{2}\lambda_s(\operatorname{tr}\boldsymbol{\epsilon})^2, \\ \boldsymbol{\tau} &\equiv \frac{\partial \mathcal{W}_s}{\partial \boldsymbol{\epsilon}} = \mu_s\boldsymbol{\epsilon} + \lambda_s\mathbf{A}\operatorname{tr}\boldsymbol{\epsilon}, \\ \boldsymbol{\epsilon} &= \frac{1}{2}\left((\nabla_s\mathbf{u})\cdot\mathbf{A} + \mathbf{A}\cdot(\nabla_s\mathbf{u})^T\right), \end{aligned} \tag{4}$$

where λ_s and μ_s are the surface elastic moduli called also surface Lamé moduli, ∇_s is the surface nabla operator, $\mathbf{A} \equiv \mathbf{I} - \mathbf{n}\otimes\mathbf{n}$ are the surface unit second-order tensor, $\mathbf{n}$ is the unit vector of outer normal to A_s, the symbol $\otimes$ designates the tensorial product between two vectors and ϵ is the infinitesimal deformations associated with the surface. In addition, we take into account the mass density associated with the surface where surface stresses are defined. This assumption results in the following formula for surface kinetic energy density [5]

$$\mathcal{K}_s = \frac{1}{2}m\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}|_{\mathbf{x}\in A_s}, \tag{5}$$

where m is the surface mass density.

For the interface, we propose the following models. We distinguish the perfect interface without discontinuities in displacements and non-perfect one when such discontinuity may exist. We attribute to the non-perfect interface two displacement fields that is one-sided limits $\mathbf{u}^- = \lim \mathbf{u}(\mathbf{x}^-, t)_{\mathbf{x}^-\to I}$ and $\mathbf{u}^+ = \lim \mathbf{u}(\mathbf{x}^+, t)_{\mathbf{x}^+\to I}$, where $\mathbf{x}^\pm \in V^\pm$. The surface strain energy density and the surface kinetic energy density are assumed to be

$$\begin{aligned} \mathcal{W}_i = {} & \mu_s^-\boldsymbol{\epsilon}^- : \boldsymbol{\epsilon}^- + \frac{1}{2}\lambda_s^-(\operatorname{tr}\boldsymbol{\epsilon}^-)^2 + \mu_s^+\boldsymbol{\epsilon}^+ : \boldsymbol{\epsilon}^+ + \frac{1}{2}\lambda_s^+(\operatorname{tr}\boldsymbol{\epsilon}^+)^2 \\ & + \frac{1}{2}[\![\mathbf{u}]\!]\cdot\mathbf{K}\cdot[\![\mathbf{u}]\!] + \frac{1}{2}[\![\nabla_s\mathbf{u}]\!] : \bar{\mathbf{K}} : [\![\nabla_s\mathbf{u}]\!], \end{aligned} \tag{6}$$

$$\mathcal{K}_i = \frac{1}{2}(m^-\dot{\mathbf{u}}^-\cdot\dot{\mathbf{u}}^- + m^+\dot{\mathbf{u}}^+\cdot\dot{\mathbf{u}}^+). \tag{7}$$

The model described by the relations (6) and (7) takes into account surface elasticity according to the Gurtin–Murdoch model, so we have two sets of surface elastic moduli $\lambda_s^\pm$ and $\mu_s^\pm$, and the adhesion (interaction) energy described by second-order tensor $\mathbf{K}$ and fourth-order tensor $\bar{\mathbf{K}}$. Tensors $\mathbf{K}$ and $\bar{\mathbf{K}}$ describes the changes of energy for $[\![\mathbf{u}]\!] \neq 0$. The square brackets denote here the jump across the interface that is $[\![\mathbf{u}]\!] = \mathbf{u}^- - \mathbf{u}^+$. $\mathbf{K}$ is similar to stiffness of the Winkler elastic foundation while $\bar{\mathbf{K}}$ relates with the