Mechanics Research Communications 74 (2016) 34–38

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

Interior formulation of axisymmetric Levinson plate theory

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ARTICLE INFO

Article history:
Received 4 July 2015
Received in revised form 6 November 2015
Accepted 29 March 2016
Available online 1 April 2016

Keywords:
Levinson theory
Saint Venant's principle
Interior plate
Clapeyron's theorem
Finite element

ABSTRACT

In this study, we show that the axisymmetric Levinson plate theory is exclusively an interior theory and we provide a consistent variational formulation for it. First, we discuss an annular Levinson plate according to a vectorial formulation. The boundary layer of the plate is not modeled and, thus, the interior stresses acting as surface tractions do work on the lateral edges of the plate. This feature is confirmed energetically by the Clapeyron's theorem. The variational formulation is carried out for the annular Levinson plate by employing the principle of virtual displacements. As a novel contribution, the formulation includes the external virtual work done by the tractions based on the interior stresses on the inner and outer lateral edges of the Levinson plate. The obtained plate equations are consistent with the vectorially derived Levinson equations. Finally, we develop an exact plate finite element both by a force-based method and from the total potential energy of the Levinson plate.

1. Introduction

The plate theories by Levinson and Reddy are widely known in the literature [1,2]. Although these theories are based on the same assumed displacement field, their governing equations are somewhat different. The Levinson theory is derived using a vectorial approach, whereas the Reddy theory is obtained by a variational formulation. The Reddy theory includes higher-order load resultants which are not present in the Levinson theory. Consequently, the Levinson theory is considered to be "variationally inconsistent" due to this difference between the two theories. In this study, we show that the Levinson theory is in fact variationally consistent with certain provisions. Our scope is limited to axisymmetric annular and circular plates.

An early investigation on a dynamic axisymmetric Levinson plate was conducted by Hutchinson [3]. An extensive study on static axisymmetric Levinson plates was carried out by Wang et al. [4], who noted the "variational inconsistency" of the Levinson plate theory but also acknowledged that the theory has some merits. The merit that really counts is the simplicity of the Levinson theory. Due to this highly appealing feature, annular and circular Levinson plates have been later studied also by other authors [5,6].

In a recent paper, we showed in an exact interior context that the displacement field on which the Levinson and Reddy beam and plate theories are based is exclusively an interior field [7]. This issue is generally left undiscussed in relation to the Levinson and Reddy theories. The use of interior kinematics means that the edge effects of a plate that decay with distance from the lateral edges of the plate are neglected by virtue of the Saint Venant's principle. Note that, for example, the well-known exact elasticity solution for a uniformly loaded simply-supported axisymmetric plate is, in fact, an interior solution (see, e.g. [8]).

In another paper [9], we showed that the Levinson beam theory [10] is an interior theory and we provided a consistent variational formulation for it. The formulation was carried out by making use of the fact that, due to the interior nature of the assumed displacement field, the stresses of the beam act as surface tractions on the lateral surfaces of the Levinson beam. In the present study, we introduce a variational formulation for the Levinson plate theory which relies on similar reasoning.

The remainder of this study is organized as follows. In Section 2, the static axisymmetric annular Levinson plate theory and its consistency with the Clapeyron's theorem are considered. In Section 3, a consistent variational formulation for the annular Levinson plate is carried out. An exact Levinson plate finite element is developed in Section 4 and conclusions are drawn in Section 5.

2. Levinson plate theory

2.1. Boundary conditions and displacement field

Fig. 1 presents an axisymmetric annular plate subjected to a rotationally symmetric transverse load $q(r)$. The thickness of the

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http://dx.doi.org/10.1016/j.mechrescom.2016.03.008

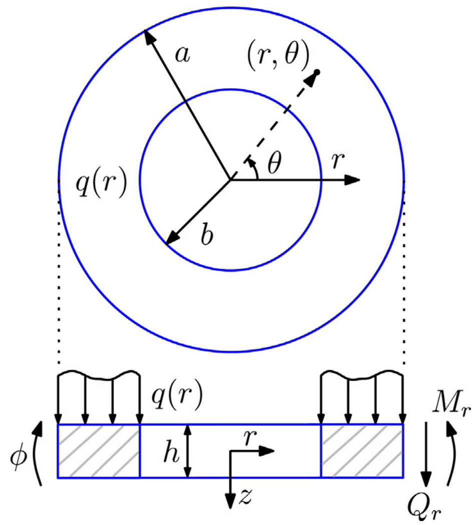

Fig. 1. Axisymmetric annular Levinson plate under a rotationally symmetric transverse load $q(r)$. The positive directions of the load resultants and the rotation ϕ on the outer edge are shown.

plate is h and the outer and inner radii of the plate are a and b, respectively. For the Levinson theory, it is assumed that (1) the transverse normal stress σ_z is zero throughout the plate and (2) the Poisson effect (lateral contraction/extension) is not present. On the basis of these assumptions, and to satisfy the homogeneous boundary conditions $\tau_{rz}(r, \pm h/2) = 0$ on the upper and lower surfaces of the plate, the displacement field can be found as [3]

$$U_r(r,z) = z\phi - \frac{4z^3}{3h^2}\left(\phi + \frac{\partial u_z}{\partial r}\right), \quad U_z(r,z) = u_z \tag{1}$$

where $u_z(r)$ is the transverse deflection of the mid-surface of the plate and $\phi(r)$ is the rotation of the normal of the mid-surface. The homogeneous stress boundary conditions are satisfied in a *strong* (pointwise) sense on the upper and lower surfaces of the plate. It is important to note that in the Levinson theory the tractions on the lateral inner and outer plate edges are not specified at each point but only through load resultants and, thus, the boundary conditions are imposed only in a *weak* sense [11]. The replacement of the true stress boundary conditions at the plate edges by the statically equivalent weak boundary conditions (load resultants) means that the exponentially decaying edge effects are neglected by virtue of the Saint Venant's principle. The cross-sectional load resultants per unit length are calculated from the equations

$$M_r(r) = \int_{-h/2}^{h/2} \sigma_r z dz, \quad M_\theta(r) = \int_{-h/2}^{h/2} \sigma_\theta z dz, \quad Q_r(r) = \int_{-h/2}^{h/2} \tau_{rz} dz. \tag{2}$$

The chosen positive directions of the load resultants $M_r(r)$ and $Q_r(r)$ are given in Fig. 1.

2.2. Implications of the interior definition

The displacement field (1) of the Levinson plate is an adequate interior field. The modeling of the boundary layer displacements (edge effects) would essentially require the use of Papkovich–Fadle-type eigenfunctions [11]. To further elucidate the interior plate definition and its implications, let us consider the complete circular plate of radius a' shown in Fig. 2. The real stress or displacement boundary conditions are given at $r = a'$. The detailed distribution of the boundary stresses would bring about edge effects which decay exponentially towards the interior plate. In other words, the edge effects are significant only in the boundary layer, the radial thickness of which is typically of the same order with the thickness of the plate. Beyond that the fully-developed interior plate solution prevails. In engineering applications, instead of using a complete plate, an interior theory is usually applied and the conditions at the interior plate edge at $r = a$ are chosen so as to imitate the true boundary conditions. This long-standing practice is well-suited especially for thin isotropic plates which are modeled using the Kirchhoff plate theory. The thinner the plate is, the weaker the edge effects are.

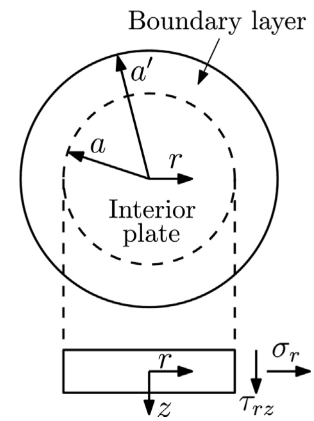

Fig. 2. Circular plate consisting of an interior plate (Levinson plate) and a boundary layer. When only the interior plate is studied, the stresses σ_r and τ_{rz} do work on the plate edge.

In the foregoing, a circular plate was discussed. In the case of an annular plate an analogous discussion may be extended to the inner boundary region. In terms of energetical considerations, the key feature of the interior plate definition is that the fully-developed interior stresses do work on the lateral edges of the plate. This has an effect on the total potential energy of the interior plate, which can be written as

$$\Pi = U + W_s \tag{3}$$

where the strain energy for an annular plate is

$$U = \pi \int_b^a \int_{-h/2}^{h/2} r(\sigma_r \epsilon_r + \sigma_\theta \epsilon_\theta + \tau_{rz}\gamma_{rz}) dz dr \tag{4}$$

and the work by surface tractions due to the interior stresses on the inner and outer lateral edges of the interior plate is given by

$$\begin{aligned}\frac{W_s}{2\pi} &= a\int_{-h/2}^{h/2} \sigma_r(a,z)U_r(a,z)dz - b\int_{-h/2}^{h/2} \sigma_r(b,z)U_r(b,z)dz \\ &\quad + a\int_{-h/2}^{h/2} \tau_{rz}(a,z)U_z(a,z)dz - b\int_{-h/2}^{h/2} \tau_{rz}(b,z)U_z(b,z)dz.\end{aligned} \tag{5}$$

This feature will be exploited in the following sections in all energy-based developments related to the Levinson plate theory.

2.3. General static solution

The vectorially derived equilibrium equations for the annular Levinson plate are [4]

$$\frac{2Gh}{3}\frac{\partial}{\partial r}\left[r\left(\phi + \frac{\partial u_z}{\partial r}\right)\right] = -rq(r), \tag{6}$$

$$\frac{D}{5}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(4r\phi - r\frac{\partial u_z}{\partial r}\right)\right]\right\} = -rq(r). \tag{7}$$

where $D = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity and E, G and ν are the Young's modulus, shear modulus and Poisson ratio,