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On a partially debonded rigid line inclusion penetrating a circular inhomogeneity

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ABSTRACT

We derive closed-form solutions to the mixed boundary value problem of a partially debonded rigid line inclusion penetrating a circular elastic inhomogeneity under antiplane shear deformation. The two tips of the rigid line inclusion are just mutual mirror images with respect to the inhomogeneity/matrix interface, and the upper part of the rigid line inclusion is debonded from the surrounding materials. By using conformal mapping and the method of image, closed-form solutions are derived for three loading cases: (i) the matrix is subjected to remote uniform stresses; (ii) the matrix is subjected to a line force and a screw dislocation; and (iii) the inhomogeneity is subjected to a line force and a screw dislocation. In the mapped ξ-plane, the solutions for all the three loading cases are interpreted in terms of image singularities. For the remote loading case, explicit full-field expressions of all the field variables such as displacement, stress function and stresses are obtained. Also derived is the near tip asymptotic elastic field governed by two generalized stress intensity factors. The generalized stress intensity factors for all the three loading cases are derived.

1. Introduction

The method of image has been widely adopted in deriving Green's function solutions in electrostatics [1] and elastostatics [2]. Usually the feasible configurations, in which the method of image can be applied to derive closed-form solutions, are restricted to those in which there exists only a single straight or circular boundary [2]. When multiple boundaries exist in a structure, the method of image will lose some attraction because now an infinite number of images are required to (approximately) satisfy the boundary conditions [3,4]. As a result the obtained Green's functions are in series forms and are approximate in nature. Some exceptions in which the number of images can still be finite even when there exist multiple boundaries include: (i) a circular conductor partially merged in a dielectric cylinder [1]; (ii) an orthotropic (including isotropic) quarter plane under antiplane deformations [5]. By using conformal mapping together with the method of image, closed-form solutions for more complex configurations with multiple boundaries are expected to be obtained. For example, in our recent study [6], we derived closed-form solutions to the problem of a finite crack partially penetrating a circular inhomogeneity under antiplane deformations. The traction-free boundary condition on the upper and lower crack surfaces is the so-called Neumann (or second-type) boundary condition (in terms of the out-of-plane displacement). Apparently closed-forms solutions can still be derived if we replace the finite crack studied in [6] by a rigid line inclusion (or stiffener, or anti-crack) with vanishing thickness, on the two surfaces of which the displacement is fixed (see the results in the Appendix). The fixed displacement condition is the Dirichlet (or first-type) boundary condition. Most recently experimental and theoretical investigations on rigid line inclusions have rekindled researchers' interest (see for example [7–9]). The objective of this

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research is to demonstrate that closed-form and elegant solutions can be derived to the mixed boundary value problem of a partially debonded rigid line inclusion penetrating a circular elastic inhomogeneity under antiplane shear deformation. More specifically the upper part of the rigid line inclusion is debonded from the surrounding materials, whereas the lower part of the line inclusion is perfectly bonded to the surrounding materials.

This work is structured as follows. In Section 2, basic formulation including the introduced conformal mapping is given. In Section 3 we study in detail the mixed boundary value problem. Three loading cases are studied in this section: (i) the matrix is subjected to remote uniform stresses (Section 3.1); (ii) the matrix is subjected to a line force and a screw dislocation (Section 3.2); and (iii) the inhomogeneity is subjected to a line force and a screw dislocation (Section 3.3). An extreme loading case of the so-called Zener–Stroh debonded anticrack is considered in Section 3.4. The obtained solutions for all these three loading cases are easily interpreted in terms of image singularities in the mapped ξ-plane. In addition the generalized stress intensity factors for all these loading cases are rigorously derived.

2. Formulation

We establish a cartesian coordinate system (x,y) and consider the two-dimensional problem of a circular elastic inhomogeneity S_1 of unit radius bonded to an infinite matrix S_2 through a sharp perfect interface L, as shown in Fig. 1. We further assume that a straight rigid line inclusion lies on the segment $x \in [1/a, a]$ and $y = 0$, where $a > 1$. Apparently the two tips of the rigid line inclusion are just mutual mirror images with respect to the circular interface $|z| = 1$. In addition the upper part of the rigid line inclusion is debonded from the surrounding materials, whilst the lower part of the rigid line inclusion is still perfectly bonded to the surrounding materials. In what follows the subscripts 1 and 2 (or the superscripts (1) and (2) for the stress components) will refer to S_1 and S_2. We will discuss the problem in anti-plane shear. Under anti-plane shear deformation, the out-of-plane displacement w, the stress function ϕ, and the stress components σ_{zy} and σ_{zx} can be expressed in terms of an analytic function $f(z)$ of the complex variable $z = x + iy = r\exp(i\theta)$ as

$$\mu^{-1}\phi + iw = f(z), \quad \sigma_{zy} + i\sigma_{zx} = \mu f'(z), \tag{1}$$

where the two stress components can be expressed in terms of the stress function as

$$\sigma_{zy} = \phi_{,x}, \quad \sigma_{zx} = -\phi_{,y}. \tag{2}$$

Apparently there are in total three boundary or interface conditions to be addressed: (i) $\phi=0$ at $x \in [1/a, a]$ and $y = 0^+$, (ii) $w=0$ at $x \in [1/a, a]$ and $y = 0^-$, (iii) $\phi_1 = \phi_2$ and $w_1 = w_2$ at $x^2 + y^2 = 1$. At first sight it seems impossible to derive a closed-form solution to this problem due to the existence of too many boundary or interface conditions to be satisfied.
Now we introduce the following conformal mapping function

$$z = \omega(\xi) = \frac{a\xi^4 + 1}{\xi^4 + a}, \quad \xi(z) = \left(\frac{az - 1}{a - z}\right)^{\frac{1}{4}}, \quad (Re\{\xi\}, Im\{\xi\} \geqslant 0) \tag{3}$$

which maps the circular inhomogeneity onto a quarter circular region: $|\xi| \leqslant 1$ and $Re\{\xi\},\ Im\{\xi\} \geqslant 0$, and maps the matrix onto $|\xi| \geqslant 1$ and $Re\{\xi\},\ Im\{\xi\} \geqslant 0$, as shown in Fig. 2. The upper part of the segment $x \in [1/a, a]$ and $y = 0^+$ is mapped onto $Im\{\xi\} = 0,\ Re\{\xi\} \geqslant 0$, and the lower part of the segment $x \in [1/a, a]$ and $y = 0^-$ is mapped onto $Re\{\xi\} = 0,\ Im\{\xi\} \geqslant 0$ (see Fig. 2). In the ξ-plane, we have two semi-infinite straight surfaces: $Im\{\xi\} = 0,\ Re\{\xi\} \geqslant 0$ and $Re\{\xi\} = 0,\ Im\{\xi\} \geqslant 0$; and

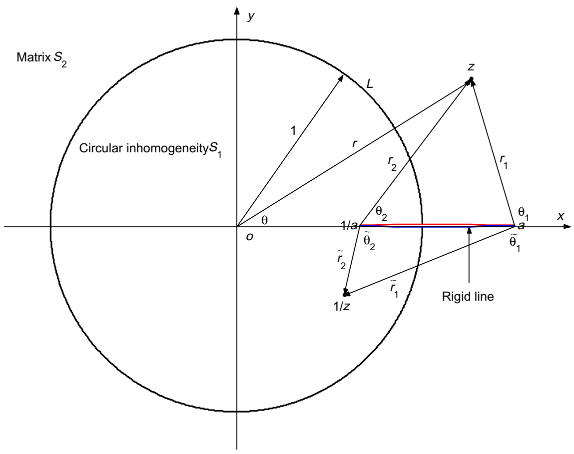

Fig. 1. A partially debonded rigid line inclusion penetrating a circular inhomogeneity. The upper part of the rigid line inclusion is debonded from the surrounding materials.