Accepted Manuscript

Positive solutions to generalized Dickman equation

Josef Diblík

PII: S0893-9659(18)30069-7
DOI: https://doi.org/10.1016/j.aml.2018.03.004
Reference: AML 5452

To appear in: *Applied Mathematics Letters*

Received date : 14 January 2018
Revised date : 5 March 2018
Accepted date : 5 March 2018

Please cite this article as: J. Diblík, Positive solutions to generalized Dickman equation, Appl. Math. Lett. (2018), https://doi.org/10.1016/j.aml.2018.03.004

Positive Solutions to Generalized Dickman Equation *†

Josef Diblík ‡§

Abstract

The paper investigates large-time behaviour of positive solutions to a generalized Dickman equation. The asymptotic behaviour of dominant and subdominant positive solutions is analyzed and a structure formula describing behaviour of all solutions is proved. A criterion is also given for sufficient conditions on initial functions to generate positive solutions with prescribed asymptotic behaviour with values of their weighted limits computed.

1 Introduction and Preliminaries

The paper investigates large-time behaviour (for $t \to \infty$) of positive solutions to a generalized Dickman equation

$$\dot{x}(t) = -\frac{\alpha}{t}\left(1-\frac{1}{t}\right)^{\alpha-1} x(t-1) \tag{1}$$

where $t \geq t_0$, t_0 is sufficiently large, $\alpha \geq 1$ (throughout the paper, it is assumed that α is fixed). If $\alpha = 1$, we get what is called Dickman (or Dickman-de Bruin) equation

$$\dot{x}(t) = -t^{-1}x(t-1). \tag{2}$$

The latter equation is important in analytic number theory since the limit $\lim_{y\to\infty} \Psi(y^t, y)y^{-t}$, $t > 0$, where $\Psi(y_1, y_2)$ is the number of positive integers not exceeding y_1 having no prime divisors exceeding y_2, called Dickman (or Dickman-de Bruin) function, is the solution of (2) generated by the unit initial function on $[0, 1]$ (we refer, for example, to [3, 7, 12]). Recently, a growing interest has been paid to the investigation of the properties of solutions to (2), for example, in [5, 13].

1.1 Basic notions

Recall some necessary basic notions. Consider an equation

$$\dot{x}(t) = -c(t)x(t-1), \tag{3}$$

which is more general than (1), where the function $c\colon [t_0, \infty) \to (0, \infty)$, $t_0 \in \mathbb{R}$ is continuous. A continuous function $x\colon [t_0-1, \infty) \to \mathbb{R}$ is called a solution of (3) on $[t_0-1, \infty)$ if it is continuously differentiable on $[t_0, \infty)$ and satisfies (3) for every $t \in [t_0, \infty)$ (at $t = t_0$, the derivative is defined as the derivative on the right). The initial problem

$$x(t) = \varphi(t), \quad t \in [t_0 - 1, t_0], \tag{4}$$

where φ is a continuous function, defines a unique solution $x = x(t_0, \varphi)(t)$, $t \geq t_0 - 1$ of (3) such that $x(t_0, \varphi)(t) = \varphi(t)$ if $t \in [t_0 - 1, t_0]$. A solution x of (3) on $[t_0 - 1, \infty)$ is called positive if

*Keywords: generalized Dickman equation; positive solution; dominant solution; subdominant solution; asymptotic behaviour; delay; weighted limit.
†AMS (MOS) subject classifications: 34K06; 34K12; 34K25
‡Brno University of Technology, Faculty of Civil Engineering, Brno, Czech Republic (diblik.j@fce.vutbr.cz)
§The author has been supported by the project No. LO1408, AdMaS UP-Advanced Materials, Structures and Technologies (supported by Ministry of Education, Youth and Sports of the Czech Republic under the National Sustainability Programme I).