Nonlinear Analysis: Hybrid Systems 31 (2019) 19–40

Contents lists available at ScienceDirect

Nonlinear Analysis: Hybrid Systems

journal homepage: www.elsevier.com/locate/nahs

Modeling and sensitivity analysis methodology for hybrid dynamical system

Sebastien Corner [a,b,*], Corina Sandu [a], Adrian Sandu [b]

[a] Department of Mechanical Engineering, Virginia Tech, Blacksburg, VA 24061, United States
[b] Computational Science Laboratory, Department of Computer Science, Virginia Tech, Blacksburg, VA 24061, United States

ARTICLE INFO

Article history:
Received 13 June 2017
Accepted 25 July 2018
Available online xxxx

Keywords:
Direct sensitivity analysis
Switching sensitivities
Constrained system
Impulsive system

ABSTRACT

This paper provides a complete mathematical framework to compute the sensitivities with respect to system parameters for any second order hybrid Ordinary Differential Equation (ODE) and ranked 1 and 3 Differential Algebraic Equation (DAE) system. The hybrid system is characterized by discontinuities in the velocity state variables due to an impulsive forces at the time of event. Such system may also exhibit at the time of event a change in the equation of motions, or in the kinematic constraints.

The methodology and the tools developed in this study compute the sensitivities of the states of the model and of the general cost functionals with respect to model parameters for both, unconstrained and constrained, hybrid mechanical systems. The analytical methodology that solves this problem is structured based on jumping conditions for both, the velocity state variables and the sensitivities matrix. The proposed analytical approach is then benchmarked against a known numerical method.

Finally, this study emphasizes the penalty formulation for modeling constrained mechanical systems since this formalism has the advantage that it incorporates the kinematic constraints inside the equation of motion, thus easing the numerical integration, works well with redundant constraints, and avoids kinematic bifurcations.

1. Introduction

Sensitivity analysis plays a key role in a wide range of computational engineering problems, such as design optimization, optimal control, and implicit time integration methods, by providing derivative information for gradient based algorithms and methods. Sensitivity analysis quantifies the effect of small changes in the system parameters onto the outputs of interest [1]. Specifically, in the design of mechanical systems, sensitivity analysis reveals the system parameters that affect the given performance criterion the most, thus providing directions for mechanical design improvements. Sensitivity analysis enables gradient-based optimization by providing the derivative of the cost function with respect to design variables. In adaptive control systems, sensitivity analysis allows assessing the stability of a system by accounting for the effects of system disturbances and system parameters inaccuracies.

The most widely used techniques for sensitivity analysis are the direct and the adjoint methods. These approaches are complementary, as the direct sensitivity provides information on how parametric uncertainties propagate through the system dynamics, while the adjoint method is suitable for inverse modeling, in the sense that it can be used to identify the origin of uncertainty in a given model output [2].

* Corresponding author at: Computational Science Laboratory, Department of Computer Science, Virginia Tech, Blacksburg, VA 24061, United States.
E-mail addresses: scorner@vt.edu (S. Corner), csandu@vt.edu (C. Sandu), sandu@cs.vt.edu (A. Sandu).

https://doi.org/10.1016/j.nahs.2018.07.003
1751-570X/

Numerical approximations of sensitivities are often calculated by finite difference methods where the deviation of the state trajectories are evaluated after system parameters disturbances or variations in the initial conditions are added to the system. Because of the simplicity of this method, which does not require any additional inputs other than the provided model, this technique is broadly used. However, the accuracy of the results is severely limited by the perturbation size and by the roundoff and cancellation errors [3].

This study develops a general and unify formulation for *direct* sensitivity analysis for hybrid dynamical system. In the context of this study, the term hybrid refers to a continuous system that encounters a finite number of events where some of the state variables jump to different values; the dynamics of a hybrid system is piecewise-smooth in time. The formalism presented and developed in this study does not support mechanisms that deal with infinite number of events and the grazing phenomenon where the trajectories of the system would make a tangential contact with the event function. For methods that take into account such a phenomenon, the reader may refer to [4–6].

We treat unconstrained systems modeled by ODEs, as well as constrained multibody systems modeled by differential algebraic equations (DAEs) and ODE penalty formulation. The penalty formulation is a constraint violation stabilization technique that incorporates all the kinematic constraints (position, velocity, and the acceleration constraint equations) into the equation of motion. A first approach of the penalty formulation was presented by Park and Chiou (1988) [7]; the penalty formulation is mentioned by the authors to be more robust and easier to implement that the Baumgarte's method. The full formulation was presented in [8]. Kurdila (1993) [9] showed the convergence and stability of the method. De Jalon and Bayo [10] were in favor on this formulation as the penalty formulation works with redundant constraints and near kinematic singular configurations, whereas the Baumgarte's method fails for such configurations . The reader may refer to the book "Kinematic and Dynamic Simulation of Multibody Systems: The Real-Time Challenge" [10] for further details on the method.

We are especially interested in hybrid mechanical systems where sensitivity analysis involves the time, the position coordinates, the velocity coordinates, and the system parameters. The sensitivity of the time of event and the jumps in the sensitivities of the state variables at the time of event, are available in the literature [11–19].

In this study, a new graphical proof of the jumps in sensitivities at the time of an event is employed, which helps to better understand the conditions for the jump in the sensitivities. This paper provides a unified methodology for determining the system solutions, their sensitivities, and sensitivities of a cost function for different types of events. The first type of event is caused by an external impulse (e.g., a contact) leading to a sudden change of velocities. The second type of event is caused by a sudden change of the equations of motion. The third type of event is caused by a sudden change in the kinematic constraints.

The paper is organized as follows. A review of the direct sensitivity approach for smooth ODE systems, along with the quadrature variable of the running cost function, is introduced in Section 2. The extension of this approach to hybrid ODE systems is presented in Section 3. The sensitivity analysis for smooth constrained rigid multibody dynamics systems is reviewed in Section 4. Sensitivity analysis methodology for hybrid constrained systems is developed in Section 5. Sensitivity analysis methodology for constrained systems with a sudden change of the equation of motions is developed in Section 6. The proposed sensitivity analysis methodologies in Section 5 are applied to a five-bar mechanism in Section 7. Conclusions are drawn in Section 8.

2. Direct sensitivity analysis for smooth ODE systems

We start the discussion with a review of direct sensitivity analysis for dynamical systems governed by smooth ODEs.

2.1. Smooth ODE systems dynamics

In this study we consider second order systems of ordinary differential equations of the form:

$$\mathsf{M}\left(t, q, \rho\right) \cdot \ddot{q} = \mathsf{F}\left(t, q, \dot{q}, \rho\right), \quad t_0 \le t \le t_F, \quad q(t_0) = q_0(\rho), \quad \dot{q}(t_0) = \dot{q}_0(\rho), \tag{1}$$

or equivalently:

$$\ddot{q} = \mathsf{M}^{-1}\left(t, q, \rho\right) \cdot \mathsf{F}\left(t, q, \dot{q}, \rho\right) =: f^{\text{eom}}\left(t, q, \dot{q}, \rho\right), \tag{2}$$

that arise from the description of the dynamics of mechanical systems. In (2) $t \in \mathbb{R}$ is time, $q \in \mathbb{R}^n$ is the generalized position vector and $\dot{q} \in \mathbb{R}^n$ is the generalized velocity vector, n is the dimension of generalized coordinates, and $\rho \in \mathbb{R}^p$ is the vector of system parameters, where p is the number of parameters. The dot notation ($\dot{\square}$ or $\ddot{\square}$) indicates the total (first or second order) derivative of a function or variable with respect to time. Subscripts indicate partial derivative with respect to a quantity, unless stated otherwise. The mass matrix $\mathsf{M} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{n\times n}$ is assumed to be smooth with respect to all its arguments, invertible, and with an inverse M^{-1} that is also smooth with respect to all arguments. The right-hand side function $\mathsf{F} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ represents external and internal generalized forces and is assumed to be smooth with respect to all its arguments.

The state trajectories are obtained by integrating the equations of motion (2), which depend on the system parameters ρ. Consequently, the state trajectories (the solutions of the equations of motion) depend implicitly on time and on the parameters, $q = q(t, \rho)$ and $\dot{q} = \dot{q}(t, \rho)$. The state trajectories also depend implicitly on the initial conditions of (2). For clarity we denote the velocity state variables by $v = \dot{q} \in \mathbb{R}^n$.