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Explicit solutions to utility maximization problems in a regime-switching market model via Laplace transforms

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HIGHLIGHTS

- A Laplace transform method for solving utility maximization problems is proposed.
- The method can be applied to solve coupled PDEs independent of the state variable.
- Rigorous verification of the optimality of the solution is provided using the theory of constrained viscosity solutions.

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ABSTRACT

We study the problem of utility maximization from terminal wealth in which an agent optimally builds her portfolio by investing in a bond and a risky asset. The asset price dynamics follow a diffusion process with regime-switching coefficients modeled by a continuous-time finite-state Markov chain. We consider an investor with a Constant Relative Risk Aversion (CRRA) utility function. We deduce the associated Hamilton–Jacobi–Bellman equation to construct the solution and the optimal trading strategy and verify optimality by showing that the value function is the unique constrained viscosity solution of the HJB equation. By means of a Laplace transform method, we show how to explicitly compute the value function and illustrate the method with the two- and three-states cases. This method is interesting in its own right and can be adapted in other applications involving hybrid systems and using other types of transforms with basic properties similar to the Laplace transform.

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1. Introduction

In this paper we study an investment problem of an agent whose portfolio is constructed by investing in a bond and a risky asset, whose price dynamics follow a diffusion process with regime-switching coefficients, modeled by an observable continuous-time finite-state Markov chain. The agent's objective is to maximize her expected utility from terminal wealth.

Changes of regime in financial markets have been empirically observed and may be due, for instance, to sudden changes in the economy or major political events. Regime-switching processes were initially proposed by Hamilton, who studied the effect of incorporating shifts in the parameters of a discrete-time model, via an unobserved discrete time two-state Markov chain, when analyzing yields on government bonds [1]. Since then, many empirical studies have argued that regime-switching modeling can help to better predict market prices behavior. More recently, Pereiro and González-Rozada [2] analyzed the market index of a sample of stock markets worldwide to test for the presence of regimes. They concluded that

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68% and 37% of the emerging and developed stock markets, respectively, show the existence of regime-switching, including the SPX in USA.

Portfolio optimization problems in continuous-time date back to the works of Merton [3,4], who proposed that the market risk is driven by a Brownian motion. In the context of regime-switching dynamics, where an auxiliary Markov process dictates the market regime, the utility maximization problem from terminal wealth has been studied under different assumptions on how regime information is available to the agent.

For partially observable regimes in which the Markov chain is hidden or not observed directly, see Sass and Haussmann [5], Nagai and Runggaldier [6] and references therein. The authors argue that explicit analytical solutions are very difficult to obtain and the optimal strategies and value function have to be determined numerically.

In the fully observable case and for an investor with logarithmic or power utilities, Capponi and Figueroa-López [7] and Fu et al. [8] consider a portfolio that contains, besides a risk-free bond and a risky stock, an extra term. In [7], the authors take into account default risk and incorporate a defaultable bond into the portfolio. They allow regime dependent short rate, drift, volatility and default intensities. By separating the problem into pre- and post default optimization subproblems, they provide the associated verification theorems for each subproblem assuming that the associated HJB equation has a smooth solution, and construct the value functions as the solution of coupled linear systems of ordinary differential equations. Fu et al. [8] consider a portfolio that also contains an option written on the stock. They approximate the value function as the limit of a sequence of value functions of auxiliary problems. More related to our paper with a portfolio built with a risk-free bond and a risky asset only, Zhang and Yin [9] consider a fairly general setup for the utility function, including the power, logarithmic and exponential functions. However, due to this generality, the HJB equation of the associated control problem is too difficult to solve explicitly. Then, they opt to tackle the problem using a singular-perturbation approach and successfully obtain *near-optimal* allocation strategies.

In this paper, we assume that the short rate, as well as the rate of return and volatility of the risky-asset depend on the market regime. The state of the market is modeled by an observable continuous-time and finite-state Markov chain. We allow the cash amount to be invested in the risky asset to be unbounded. Due to this relaxed assumption on the control set, the HJB equation associated to the maximization problem is of degenerate parabolic type. Therefore, a smooth solution V cannot be assumed to exist and the classical verification result based on an application of Itô's formula is not possible.

The notion of viscosity solutions has been successfully applied in many contexts when a smooth solution to a PDE equation is not expected to exist. For instance, in the case of stochastic controlled problems, a non-exhaustive list includes the works of Lions [10], Duffie and Zariphopoulou [11], Duffie et al. [12], Kounta [13], and others. In the case of optimal stopping problems Pemy and Zhang [14], Li [15], Bian et al. [16]. For a general overview of the theory of viscosity solutions of second order PDEs we refer to Crandall, Ishii and Lions [17], and for their connection to stochastic optimization problems we refer to Fleming and Soner [18] . We shall show that V is the unique *constrained* viscosity solution of the associated HJB equation in an appropriate class of functions, in a sense to be specified later.

One of the main contributions of this paper is that we present a simple methodology to explicitly compute the value function based on inverse Laplace transforms, which is a new idea in the literature on portfolio optimization problems. We think that this idea can be easily applied to other optimization problems involving regime-switching diffusions and is interesting in its own right.

The rest of the paper is organized as follows. The model dynamics and problem formulation is presented in Section 2. In Section 3, we heuristically construct a candidate trading (portfolio allocation) strategy and value function by martingale and dynamic programming arguments. In Section 4, we show that V is the unique *constrained* viscosity solution of the associated HJB equation, in a sense to be specified later, and a verification that the candidate solution satisfies all the necessary conditions, implying that it coincides with the value function. We also obtain optimal allocation strategies in the form of feedback controls. The Laplace transform method to compute the value function is presented in Section 5 and some numerical experiments are provided in Section 6. The final section summarizes the main results.

2. Model dynamics and problem setup

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which supports a Brownian motion $B = (B_t)_{t\geq 0}$ and a continuous-time Markov chain $Y = (Y_t)_{t\geq 0}$ with finite state space $\mathcal{M} = \{1, 2, \dots, m\}$ and generator $Q = (q_{ij})_{m\times m}$ which satisfies

$$q_{ij} \geq 0 \quad \text{for } i \neq j, \qquad \sum_{j\in\mathcal{M}} q_{ij} = 0, \qquad q_i := -q_{ii} > 0.$$

We denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the $\mathbb{P}$-augmentation of the filtration generated by B and Y. Lemma 2.5 in [19] ensures that B and Y are independent.

Let $P = (P_t)_{t\geq 0}$ and $S = (S_t)_{t\geq 0}$ denote the price of the bond and the risky asset, respectively. We assume that these processes satisfy the Markov-modulated dynamics

$$\begin{aligned} dP_t &= r(Y_t)P_t\,dt, \\ dS_t &= \mu(Y_t)S_t\,dt + \sigma(Y_t)S_t\,dB_t \end{aligned}$$

where $r(i) > 0$, $\mu(i) > 0$ and $\sigma(i) > 0$ denote the risk-free interest rate, the rate of return of the asset, and the volatility at regime i, respectively.