Acta Astronautica 146 (2018) 181–184

Contents lists available at ScienceDirect

Acta Astronautica

journal homepage: www.elsevier.com/locate/actaastro

Collocation of equilibria in gravitational field of triangular body via mass redistribution

Alexander A. Burov [a], Anna D. Guerman [b,*,1], Vasily I. Nikonov [b,2]

[a] Federal Research Center "Computer Science and Control" of Russian Academy of Sciences, National Research University "Higher School of Economics", Moscow, Russia
[b] Centre for Mechanical and Aerospace Science and Technologies, University of Beira Interior, Covilhã, Portugal

ARTICLE INFO

Keywords:
Gravitational field modelling
Triangular gravitating body
Equilibrium point
Stability

ABSTRACT

We consider a gravitating system with triangular mass distribution that can be used as approximation of gravitational field for small irregular celestial bodies. In such system, the locations of equilibrium points, that is, the points where the gravitational forces are balanced, are analyzed. The goal is to find the mass distribution which provides equilibrium in a pre-assigned location near the triangular system, and to study the stability of this equilibrium.

1. Introduction

Due to constantly growing interest to exploration of small celestial bodies, the importance of modelling complex gravitational fields increases. In such modelling, approximation of small irregular celestial bodies by gravitating bodies with triangular mass distributions can be used, making the dynamics analysis for triangular bodies especially relevant (see, e.g., [9,10,17,19–22]).

An important property of a complex non-central gravitational field is possible existence of critical points in which the gravitational forces are balanced; such positions are called equilibria. One can consider both direct and inverse problems for existence of such equilibria. Within the framework of the direct problem, the mass distribution is given, and the equilibria are to be found. In Ref. [14] non-trivial critical points of the gravitational field are found for a regular triangle with equal masses in its vertices (see also [2,12,13]). When the degree of instability of the central solution in the planar problem is equal to two, and the gravitational potential at this point reaches a local maximum, the three detected non-trivial equilibria are of the saddle type with degrees of instability equal to one. For the inverse problem, the purpose is to find the mass distributions that correspond to a pre-assigned equilibrium.

In the present paper, the inverse equilibrium problem is discussed. The mass distributions for a triangular rigid body are found depending on the location of the equilibrium points; stability of these equilibria is studied.

2. Statement of the problem

Consider three homogeneous balls $\mathscr{G}_1$, $\mathscr{G}_2$, and $\mathscr{G}_3$, defined by their centres Q_1, Q_2, and Q_3, fixed in the absolute space, radii R_1, R_2, and R_3, and masses m_1, m_2, and m_3 respectively; the balls don't intersect each other. The points Q_1, Q_2, and Q_3 belong to the plane Π which is a symmetry plane of the system. The point Q of mass m moves in the gravitational field of the above balls. Denote

$$\mathbf{r}_i = \overrightarrow{Q_iQ}, \quad \rho_i = (\mathbf{r}_i, \mathbf{r}_i)^{1/2}, \quad M = m_1 + m_2 + m_3 \neq 0, \quad \mu_i = m_i/M,$$

then the potential energy of Newtonian attraction reads

$$U_G = mMGU, \quad U = U_1 + U_2 + U_3, \tag{1}$$

$$U_i = \mu_i U_i^{'}, \quad U_i^{'} = \begin{cases} -\dfrac{1}{\rho_i}, & \rho_i \geq R_i \\ \dfrac{\rho_i^2 - 3R_i^2}{2R_i^3}, & \rho_i < R_i \end{cases} \tag{2}$$

The first expression in (2) corresponds to the location of the point Q outside the ball $\mathscr{G}_i$; the second one corresponds to the position of Q inside $\mathscr{G}_i$ (Fig. 1).

At first glance, examination of motion inside a body makes little sense; however "penetrable" bodies are used for modelling of dust clouds

* Corresponding author.
E-mail addresses: teormech@gmail.com (A.A. Burov), anna@ubi.pt (A.D. Guerman), nikon_v@list.ru (V.I. Nikonov).
[1] IAA Member, Section 2.
[2] On leave from FRC "Computer Science and Control" of the Russian Academy of Sciences.

https://doi.org/10.1016/j.actaastro.2018.02.032
Received 20 February 2018; Accepted 21 February 2018
Available online 3 March 2018

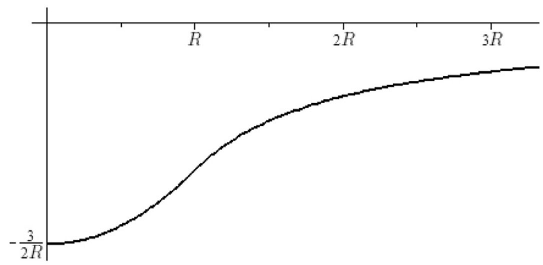

Fig. 1. Gravitational potential of a penetrable homogeneous ball (2).

and galaxies (see, e.g. [8]).

To study existence and stability of equilibria, the first and second derivatives of functions (2) have to be calculated. Let *Ozxy* be a reference frame (RF) fixed in the triangle; the plane *Ozx* coincides with the plane *Π* Then

$$\overrightarrow{OQ}_i = \mathbf{q}_i = (z_i, x_i, 0)^T, \quad \overrightarrow{OQ} = \mathbf{q} = (z, x, y)^T,$$

and the expressions for derivatives are

$$\frac{\partial U_i'}{\partial z} = \begin{cases} \dfrac{z - z_i}{\rho_i^3}, & \rho_i \geq R_i \\ \dfrac{z - z_i}{R_i^3}, & \rho_i < R_i \end{cases} \quad (z, x, y) \tag{3}$$

$$\frac{\partial^2 U_i'}{\partial z^2} = \begin{cases} \dfrac{\rho_i^2 - 3(z - z_i)^2}{\rho_i^5}, & \rho_i \geq R_i \\ \dfrac{1}{R_i^3}, & \rho_i < R_i \end{cases} \quad (z, x, y) \tag{4}$$

$$\frac{\partial^2 U_i'}{\partial z \partial x} = \begin{cases} -3\dfrac{(z - z_i)(x - x_i)}{\rho_i^5}, & \rho_i \geq R_i \\ 0, & \rho_i < R_i \end{cases} \quad (z, x, y) \tag{5}$$

where notation (z, x, y) denotes cyclical substitution of symbols z, x and y.

Using (3), equilibria are found from the following system

$$\frac{\partial U}{\partial p} = 0, \quad p \in \{z, \ x, \ y\}. \tag{6}$$

Since Π is the plane of symmetry, the third equation of system (6) possesses solution $y = 0$. Later on, we consider only these solutions (z, x, 0); they are located in the plane Π. Taking into account the first and the second equations from (6), one can write down equilibrium equations as

$$y = 0, \quad \mu_1 \frac{\partial U_1'}{\partial p} + \mu_2 \frac{\partial U_2'}{\partial p} + \mu_3 \frac{\partial U_3'}{\partial p} = 0 \quad p \in \{z, \ x\}. \tag{7}$$

3. Solution to the inverse problem

Let us choose a point Q_0 $(z_0, x_0, 0)$ on the plane Π and find mass parameters μ_i, $i = 1, 2, 3$, such as Q_0 is an equilibrium. System (7) together with relation

$$\mu_1 + \mu_2 + \mu_3 = 1 \tag{8}$$

is linear with respect to μ_1, μ_2, μ_3. Using Cramer's rule, the solution of this system can be represented as

$$\mu_i = \frac{D_i}{D}, \quad i = 1, 2, 3 \tag{9}$$

$$D_1 = \frac{\partial U_2'}{\partial z}\frac{\partial U_3'}{\partial x} - \frac{\partial U_2'}{\partial x}\frac{\partial U_3'}{\partial z} \quad (1, 2, 3)$$

$$D = \frac{\partial U_1'}{\partial z}\left(\frac{\partial U_2'}{\partial x} - \frac{\partial U_3'}{\partial x}\right) + \frac{\partial U_2'}{\partial z}\left(\frac{\partial U_3'}{\partial x} - \frac{\partial U_1'}{\partial x}\right) + \frac{\partial U_3'}{\partial z}\left(\frac{\partial U_1'}{\partial x} - \frac{\partial U_2'}{\partial x}\right),$$

where $(1, 2, 3)$ denotes cyclical substitution of the respective indices. If $D \neq 0$, the solution of the inverse problem is found: for mass parameters μ_i, $i = 1, 2, 3$ from (9) $Q_0 = (z_0, x_0, 0)$ is an equilibrium. Meanwhile, the sign of μ_i, $i = 1, 2, 3$ is not necessarily positive. For instance, if one assumes that the radii of the balls are very small, then, to provide equilibria at points located outside the triangle $Q_1Q_2Q_3$, some negative masses are required, which might appear strange. However, negative masses are applied in modelling of gravitational fields for bodies of complex shape, for example, to compensate for cavities or some volumes counted twice (see, e.g. [1]). Moreover, mechanics of systems with negative masses, both gravitational and inert, is the subject of active study in modern celestial mechanics [3–5,15] and physics [18].

4. Degree of instability

The degree of instability for the above equilibria can be found by examination of signs for the eigenvalues of Hessian for potential (1) (see, e.g. [6,11,16]). The respective characteristic polynomial reads

$$P'(\sigma) = -\sigma^3 + p_1\sigma^2 - p_2\sigma + p_3 = \det\begin{pmatrix} U_{zz} - \sigma & U_{zx} & U_{zy} \\ U_{xz} & U_{xx} - \sigma & U_{xy} \\ U_{yz} & U_{yx} & U_{yy} - \sigma \end{pmatrix}. \tag{10}$$

Here $U_{sp} = \frac{\partial^2 U}{\partial s \partial p}$, s, $p \in \{z, x, y\}$; the derivatives are calculated on the intended equilibrium $Q_0 = (z_0, x_0, 0)$.

Calculations show that for $y = 0$ some derivatives vanish, namely, $U_{zy} = U_{yz} = U_{xy} = U_{yx} = 0$, and the polynomial $P'(\sigma)$ takes the form

$$\begin{aligned} &P'(\sigma) = (\sigma_1 - \sigma)P_2(\sigma), \\ &P_2(\sigma) = \sigma^2 - c_1\sigma + c_2, \\ &\qquad \sigma_1 = U_{yy} \end{aligned} \tag{11}$$

$$c_1 = U_{xx} + U_{zz}, \quad c_2 = U_{xx}U_{zz} - U_{zx}U_{xz}. \tag{12}$$

Taking into account expressions (4), the value $\sigma_1 = U_{yy}$ in the expanded form can be written as

$$\sigma_1 = \begin{cases} \dfrac{\mu_1}{\rho_1^3} + \dfrac{\mu_2}{\rho_2^3} + \dfrac{\mu_3}{\rho_3^3}, & \rho_i \geq R_i, \quad i = 1, 2, 3, \\ \dfrac{\mu_1}{R_1^3} + \dfrac{\mu_2}{\rho_2^3} + \dfrac{\mu_3}{\rho_3^3}, & \rho_1 < R_1, \\ \dfrac{\mu_1}{\rho_1^3} + \dfrac{\mu_2}{R_2^3} + \dfrac{\mu_3}{\rho_3^3}, & \rho_2 < R_2, \\ \dfrac{\mu_1}{\rho_1^3} + \dfrac{\mu_2}{\rho_2^3} + \dfrac{\mu_3}{R_3^3}, & \rho_3 < R_3, \end{cases}$$

If $\mu_i \geq 0$, $i = 1, 2, 3$, the value of σ_1 is non-negative. Thus, the problem of studying the degree of instability is reduced to investigation of signs σ_2, σ_3 of the polynomial $P_2(\sigma)$. Note, that

$$\sigma_1 + \sigma_2 + \sigma_3 = U_{yy} + U_{xx} + U_{zz}.$$

This sum is always zero outside the balls, so there are always values of opposite signs among σ_1, σ_2, and σ_3. Therefore, the equilibrium is unstable ("Earnshaw's theorem" [7]), and the degree of instability χ is either one or two: $\chi = 1$ or $\chi = 2$. (Note that inside any of the balls this is not the case.)

Let us consider the equilibrium points outside the balls. Assume $\sigma_1 > 0$. Then

$$c_1 = \sigma_2 + \sigma_3 < 0, \tag{13}$$