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The 'damping effect' in the dynamic response of stochastic oscillators

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ABSTRACT

We consider the dynamics of linear damped oscillators with stochastically perturbed natural frequencies. When average dynamic response is considered, it is observed that stochastic perturbation in the natural frequency manifests as an increase of the effective damping of the system. Assuming uniform distribution of the natural frequency, a closed-from expression of equivalent damping for the mean response has been derived to explain the 'increasing damping' behaviour. In addition to this qualitative analysis, a comprehensive quantitative analysis is proposed to calculate the statistics of frequency response functions from the probability density functions of the natural frequencies. Firstly, single-degree-of-freedom-systems are considered and closed-form analytical expressions for the mean and variance are obtained using a hybrid Laplace's method. Several probability density functions, including gamma, normal and lognormal distributions, are considered for the derivation of the analytical expressions. The method is extended to calculate the mean and the variance of the frequency response function of multiple-degrees-of-freedom dynamic systems. Proportional damping is assumed and the eigenvalues are considered to be independent. Results are derived for several probability density functions and damping factors. The accuracy of the approach for both single and multiple-degrees-of-freedom systems is examined using the direct Monte Carlo simulation.

1. Introduction

Damped linear oscillators have been used to model a range of physical problems across different length and time scales, and disciplines including engineering, biology and nanotechnology. Examples include nanoscale oscillators used as ultra sensitive sensors [1], vibration of buildings and bridges under earthquake loads, vibration of automobiles and aircrafts. The equation of motion of a damped oscillator can be expressed as

$$m\ddot{u}(t) + c\dot{u}(\tau) + ku(t) = f(t) \quad (1)$$

where t, $u(t)$, m, c, k are respectively the time, displacement, mass, damping, stiffness and applied forcing. Diving by m, this equation can be expressed as

$$\ddot{u}(t) + 2\zeta_n\omega_n\dot{u}(\tau) + \omega_n^2 u(t) = f(t)/m \quad (2)$$

where $\omega_n = \sqrt{k/m}$ is the natural frequency and $\zeta_n = c/2\sqrt{km}$ is the damping ratio. A rich body of literature on random vibration [2,3] is available for the case when the forcing function is random in nature. We are interested in understanding the motion when the natural frequency of the system is perturbed in a stochastic manner.

Uncertainty in the natural frequency can arise in uncertainties in the stiffness or inertia properties of the structure. These can be attributed to stochastic parametric variation in the Young's modulus, Poisson's ratio, density, or geometry of the system. In general, stochastic finite element based methods (for example, [4–8]) are well suited to deal with problems with random (distributed) parameters. For a single degree of freedom (SDOF) system, the dynamic response due to uncertainties in the natural frequency can be easily obtained using Monte Carlo simulation. Such an approach, however, may not shed light into the nature of the response statistics to be discussed in the paper. The use of reduced computational methods such as perturbation method or polynomial chaos [9] works well in general except when response near the resonance frequency is considered [10]. From an engineering point of view, this is exactly where a reliable estimate of dynamic response is necessary as this is crucial to safe design of dynamic structures.

This paper gives an explanation as to why mean based analytical approximations (e.g., perturbation, polynomial chaos) fail to provide accurate statistical description of the dynamics response near the resonance frequency of a damped system. In Section 2 some simulation results are provided as the motivation of this study. Based on this, few key observations are made and an explanation based on the mean response for the case of uniform distribution of the natural frequency is provided in Section 3. A

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quantitative analytical approach for dynamic response statistics of single-degree-of-freedom (SDOF) systems is presented in Section 4. The calculation of the probability density function (pdf) of the response is outlined in Section 4.1 and the expressions for the mean and standard deviation are derived in Section 4.2. These expressions depend on the calculation of three integrals, which are evaluated through Laplace's method and through a proposed modified Laplace's method in Sections 4.2.1 and 4.2.2. Exact expressions of the mean and standard deviation are obtained for the uniform distribution of eigenvalues in Section 5.1. Laplace's method and modified Laplace's method are developed for normal, gamma and lognormal distributions respectively in Sections 5.2, 5.3 and 5.4. The method is extended to obtain mean and standard deviation of the response for multiple-degree-of-freedom (MDOF) systems in Section 6. A numerical example for a MDOF system is shown in Section 6.4, where the proposed methods are compared to MCS. The main results and the key conclusions arising from this study are discussed in Sections 7 and 8.

2. Dynamic response of damped stochastic oscillators

2.1. Uncertainty model

Suppose the natural frequency is expressed as $\omega_n^2 = \omega_{n_0}^2 x$, where ω_{n_0} is the deterministic frequency and x is a random variable with a given probability distribution function. We assume that the mean of x is 1 and the standard deviation is σ. Stochastic perturbation of this kind can represent statistical scatter of measured values or a lack of knowledge regarding the natural frequency. Of course in the special case when the standard deviation of the random variable is close to zero, the stochastic oscillator approaches the classical deterministic oscillator. For initial simulation results, three different types of random variables, namely uniform, normal and lognormal, are considered as shown in Fig. 1.

Note that normal random variable is not a good choice for a positive quantity as the squared natural frequency. It is kept here only for comparing the results later.

2.2. Dynamic response in the time and frequency domain

Dynamic response of a SDOF system with initial displacement u_0 and initial velocity v_0 can be obtained [11] using

$$u(t) = A\, e^{\zeta_n \omega_n t} \sin(\omega_d t + \phi) \tag{3}$$

where the $\omega_d = \omega_n\sqrt{1-\zeta_n^2}$ is the damped natural frequency and the amplitude and the phase of the response are

$$A = \sqrt{u_0^2 + \left(\frac{v_0 + \zeta_n\omega_n u_0}{\omega_d}\right)^2} \quad \text{and} \quad \phi = \tan^{-1}\frac{u_0\omega_d}{v_0 + \zeta_n\omega_n u_0}. \tag{4}$$

In Fig. 2 we show the deterministic and mean response of the oscillator due to an initial displacement. The time axis is scaled with the deterministic time period $T_{n0} = 2\pi/\omega_{n0}$ so that the results become general. A representative damping factor of 5%, three types of random variables and two values of standard deviations are used for illustration. Deterministic response, sample responses of the random system (with uniform distribution) and mean response due to the three cases with random natural frequencies are shown in the figure. The mean response is significantly 'damped' compared to the deterministic response. Additionally, the 'damping effect' is almost independent to the nature of the statistical distribution of the natural frequencies.

The normalised steady-state response amplitude in the frequency domain of an SDOF oscillator can be expressed as

$$\left|\frac{u}{u_{st}}\right| = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta_n r)^2}}. \tag{5}$$

Here the static deformation $u_{st} = F/k$ where F is the amplitude of the harmonic excitation and the frequency ratio $r = \omega/\omega_{n0}$. In Fig. 3, the dynamic response of the deterministic system and the mean responses due to three cases with random natural frequencies are shown. The frequency axis is scaled with the deterministic frequency ω_{n0} for generality. Like the time-domain response, we observe that the mean response is significantly more damped compared to the deterministic response. Although the mean responses for different pdfs of ${\omega_n}^2$ are slightly different, the predominant feature (i.e., the 'damping effect') is mainly depended on the standard deviation of the random variable. The observations in these results can be summarised as:

- The mean response of a SDOF oscillator with random natural frequency is more damped compared to the underlying deterministic response.
- The higher the randomness (standard deviation), the higher the 'effective damping'.

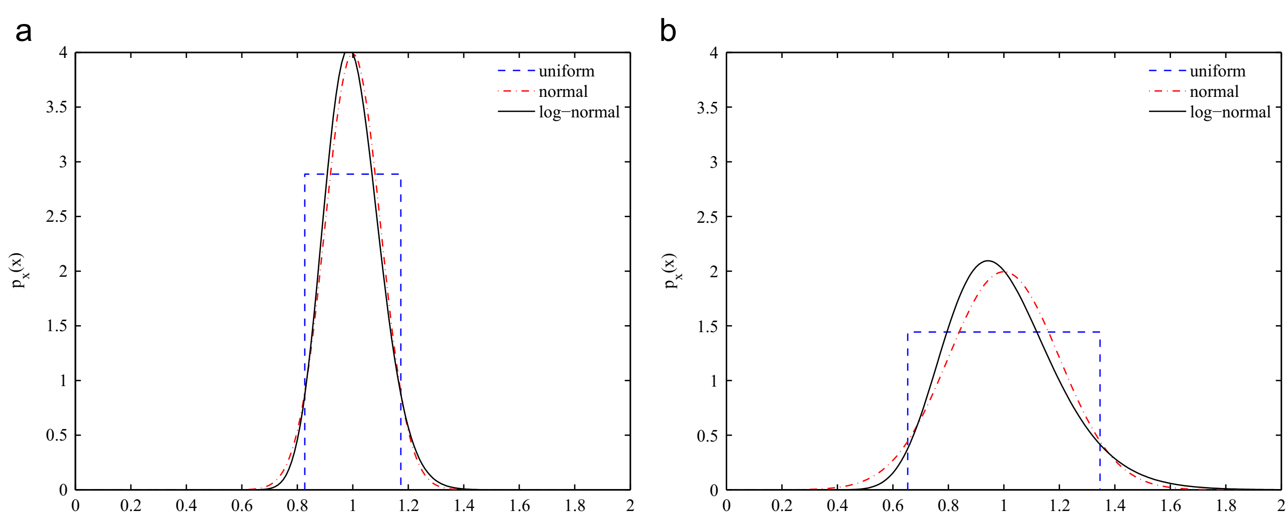

Fig. 1. Assumed probability density functions of the squared natural frequency $\omega_n^2 = \omega_{n0}^2 x$. We consider that the mean of x is 1 and the standard deviation is σ_a. (a) Pdf: $\sigma_a = 0.1$. (b) Pdf: $\sigma_a = 0.2$.