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Rare-event probability estimation with adaptive support vector regression surrogates

J.-M. Bourinet [a,b,*]

[a] SIGMA Clermont, 63178 Aubière, France
[b] CNRS UMR 6602, Institut Pascal, 63171 Aubière, France

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ABSTRACT

Assessing rare event probabilities still suffers from its computational cost despite some available methods widely accepted by researchers and engineers. For low to moderately high dimensional problems and under the assumption of a smooth limit-state function, adaptive strategies based on surrogate models represent interesting alternative solutions. This paper presents such an adaptive method based on support vector machine surrogates used in regression. The key idea is to iteratively construct surrogates which quickly explore the safe domain and focus on the limit-state surface in its final stage. Highly accurate surrogates are constructed at each iteration by minimizing an estimation of the leave-one-out error with the cross-entropy method. Additional training points are generated with the Metropolis–Hastings algorithm modified by Au and Beck and a local kernel regression is made over a subset of the known data. The efficiency of the method is tested on examples featuring various challenges: a highly curved limit-state surface at a single most probable failure point, a smooth high-dimensional limit-state surface and a parallel system.

1. Introduction

Assessing the reliability of highly safe systems is a field of great importance in many engineering applications. Despite some efficient and widely accepted methods, assessing low failure probabilities is still often too computationally demanding in real applications in which a single call to a numerical model may last minutes, hours or even days (e.g. with finite element solutions in structural mechanics). Some other challenging situations also arise when the stochastic inputs are modeled by means of a large number of random variables (e.g. with inputs modeled as random fields or random processes) and/or when the so-called limit-state surface used for defining the failure criterion(a) is characterized by a rather intricate geometry in the random space (e.g. highly curved or noisy limit-state surfaces, multiple most probable failure points of similar weights).

The scope of this work is restricted to time-invariant reliability problems such as defined in the structural reliability literature [1,2] a.k.a. static simulation problems or models by some other authors [3,4], in which time is not an explicit variable. The probability w.r.t. an undesired or unsafe state of the system of interest is expressed in terms of a n-dimensional random vector $\mathbf{X}$ of known continuous joint probability density function $f_{\mathbf{X}}$. Failure is defined in terms of a so-called limit-state function (LSF) $g : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto g(\mathbf{x})$, which can only be evaluated pointwise and where $\mathbf{x}$ represents a realization of the random vector $\mathbf{X}$. The analysis is restricted here to a single function g but this function may represent a combination of several failure modes in more general settings. The limit-state surface (LSS) $\mathcal{F}_{\mathbf{x}}^0 = \{\mathbf{x} \in \mathcal{X} : g(\mathbf{x}) = 0\}$ divides the space of realizations of the random vector $\mathbf{X}$ in a failure domain conventionally defined as $\mathcal{F}_{\mathbf{x}} = \{\mathbf{x} \in \mathcal{X} : g(\mathbf{x}) \leq 0\}$ and a safe domain defined as the complementary domain $\overline{\mathcal{F}}_{\mathbf{x}} = \{\mathbf{x} \in \mathcal{X} : g(\mathbf{x}) > 0\}$. The failure probability p_{f} therefore reads:

$$p_{\mathrm{f}} = \int_{\mathcal{F}_{\mathbf{x}}} f_{\mathbf{X}}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \mathbb{E}_{f_{\mathbf{X}}}\left[\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X})\right] \tag{1}$$

where $\mathrm{d}\mathbf{x} = \mathrm{d}x_1 \cdots \mathrm{d}x_n$ and $\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}$ is the indicator function of the failure domain $\mathcal{F}_{\mathbf{x}}$: $\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathcal{F}_{\mathbf{x}}$, $\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) = 0$ otherwise.

We will assume here that we can reformulate this problem in the so-called standard space $\mathbf{u} \in \mathbb{R}^n$, where $\mathbf{U}$ is a random vector with independent standard normal components and $\varphi_n(\mathbf{u})$ is the n-dimensional standard normal joint probability density function (pdf). This can be achieved e.g. by means of the Nataf [5] or Rosenblatt [6] transform, not recalled here for the sake of brevity.

* Correspondence address: SIGMA Clermont, 63178 Aubière, France.
Tel.: +33 4 73 28 81 16; fax: +33 4 73 28 80 27.
E-mail address: bourinet@sigma-clermont.fr

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The failure probability p_f in the standard normal space then reads:

$$p_f = \int_{\mathcal{F}_\mathbf{u}} \varphi_n(\mathbf{u})\, d\mathbf{u} = \mathbb{E}_{\varphi_n}\left[\mathbb{1}_{\mathcal{F}_\mathbf{u}}(\mathbf{U})\right] \tag{2}$$

where $\mathcal{F}_\mathbf{u} = \left\{\mathbf{u} \in \mathbb{R}^n : G(\mathbf{u}) = g(\mathbf{x}(\mathbf{u})) \le 0\right\}$ and $d\mathbf{u} = du_1, \ldots, du_n$.

The failure probability p_f in Eq. (1) or (2) can be basically assessed by two main types of methods: *sampling methods* such as the Monte Carlo method which give an estimate of p_f based on samples of the random vector $\mathbf{X}$ or *approximation methods* whose objective is to construct a surrogate of the true LSF g (or G in the standard space) based on some a priori selected assumptions and use it for the evaluation of p_f.

Among sampling methods, the crude Monte Carlo method does not make any hypothesis on the shape of the LSS but it is known for its inefficiency for estimating probabilities of rare events. Several sampling techniques are available in order to alleviate the computational expense in terms of number of calls to the LSF required for a given accuracy on the failure probability estimate. We here focus on one of them known as subset simulation [7] or adaptive multilevel splitting by some other authors [8]. It is worth mentioning that this very efficient method still requires thousands of calls to the LSF for an acceptable accuracy on p_f estimate.

Interesting alternatives to sampling methods have been proposed in an effort to still lower the computational cost but at the detriment of some predefined and restrictive assumptions. FORM and SORM techniques (see e.g. [1,2]) are examples of such methods, which consist in respectively considering the first- and second-order Taylor polynomial of G at the so-called most probable failure point (MPFP) $\mathbf{u}^*$ in the standard space. Surrogate models a.k.a. metamodels or simply response surfaces also belong to the category of approximation methods. Several types of function approximation are available including polynomial response surfaces, artificial neural networks, moving least-squares, Kriging (a.k.a. Gaussian process emulators), support vector machines (SVMs), polynomial chaos expansions among others. The common principle of these methods is to construct a surrogate model from a set of known input–output pairs referred to as design of experiments (DoE) or training set in the sequel. Several adaptive strategies have been developed in the context of reliability assessment with moving least squares [9], artificial neural networks [10,11], Kriging [12–16], SVMs [17–21], polynomial chaos expansions [22,23] among others. A common idea shared in these works is to start from an initial set of training points and enrich it by sequentially adding new training points based on a selected criterion. These criteria exploit the information gained from the surrogate model constructed at the current iteration: mean and variance of the Kriging predictor for Gaussian process emulators, distances of points of the training set to the approximate LSS for SVMs used in classification, etc. The main differences between the developed methods stem from the selected enrichment strategies although it is quite acknowledged that training points added close to the LSS contribute the most to the accuracy of the estimated failure probability. It is important to point out that the functional approximation of the selected type of surrogate must be able to capture the unknown geometry of the LSS. Due to the underlying hypotheses of the chosen surrogate model (degree and set of terms used in a polynomial response surface, type of kernel for Kriging and SVMs, etc.), this is not always possible whatever the size of the training set and, as a consequence, the failure probability assessed from the surrogate could be biased. It is however worth mentioning that some sampling-based techniques not addressed in this paper are available and may be applied to the constructed surrogate in order to correct this bias at the expense of additional calls to the LSF, such as explored e.g. in [24].

The objective of this paper is to propose an adaptive technique for assessing low failure probabilities based on SVM surrogates. The SVM model used in the present work is based on the ϵ-insensitive loss function [25] as explored in the context of reliability assessment in a few other works [26–30]. This regression approach differs from most of the works based on SVMs which consider reliability assessment as a classification problem [17,19–21,31,32]. The acronym SVR will be used for SVMs in regression, as opposed to SVC for classification. The proposed method consists in constructing a sequence of adaptive SVR surrogates $\widetilde{G}_s(\mathbf{u})$, $s = 1, \ldots, s_{\max}$ in the standard space with training points which progressively reach and populate the failure domain $\mathcal{F}_\mathbf{u}$. The new training points at each iteration s are generated from the currently constructed SVR surrogate model $\widetilde{G}_s$ by means of Monte Carlo Markov chains (MCMC) with the modified Metropolis–Hastings algorithm of Au and Beck [7]. An updated SVR surrogate model is constructed at each iteration s. This SVR is obtained by training over a subset of all the points generated from $s=1$ whose LSF values are known. Finding the most accurate surrogate from the training data at each iteration s is a central issue. In the proposed method, optimal values are obtained for the surrogate model parameters by minimizing an estimate of the leave-one-out (LOO) error proposed by Chang and Lin [33] for ϵ-insensitive SVR. This is achieved with the robust and efficient cross-entropy (CE) method [34]. The approximation of the failure probability p_f is evaluated from the SVR surrogate $\widetilde{G}_{s_{\max}}$ trained at the final iteration when a prescribed accuracy criterion is met. The proposed method is applied to three challenging problems in order to test its applicability: a highly curved LSS, a high dimensional smooth LSS and a parallel system. Results are compared with those obtained by other methods in previously published papers.

The paper is organized as follows. SVM regression by means of L1-ϵ-SVR is presented in Section 2. This section also describes the efficient stochastic search technique applied to an estimate of the generalization error in order to tune the hyperparameters of the SVR surrogates. The adaptive strategy for reliability assessment is described in Section 3. The three application examples are treated in Section 4. A conclusion is finally given in Section 5 with some important remarks and perspectives.

2. Support vector machine surrogates

Support vector machines (SVMs) are parts of statistical learning theory and the reader may refer to [35,36] for a presentation of their theoretical basis. SVMs have been successfully applied to a number of fields over the last 20 years such as handwritten digits and objects recognition, face detection, text categorization, gene selection for disease classification, etc. SVMs have been initially introduced for classification problems but later extended to regression [37]. SVMs are known for their good generalization performances and their ability to handle nonlinear models by means of kernels. The optimization problem to solve both in classification and regression for defining the SVM model is convex. This therefore guarantees a unique and global solution and allows a primal–dual interpretation. Nonlinear models assume an implicit mapping from the input space to the so-called feature space, by using a kernel in the dual optimization problem. The notations used in this section are similar to those commonly found in the SVM literature for a clear reference to existing works. In the present framework, $\mathbf{x}$ stands for a point of the standard space $\mathbf{u}$, y represents the unknown output of the LSF G at a point of interest $\mathbf{u}$, $(\mathbf{x}_i, y_i)$ for $i = 1, \ldots, N$ are the given data pairs of the training set such that $y_i = G(\mathbf{u}_i)$ and the SVR model $\widetilde{f} : \mathbf{x} \mapsto \widetilde{f}(\mathbf{x}) = f(\mathbf{x}) + b$ represents the surrogate model $\widetilde{G}_s : \mathbf{u} \mapsto \widetilde{G}_s(\mathbf{u})$ which needs to be constructed at current iteration s.