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Analytical solution of the fractional point kinetics equations with multi-group of delayed neutrons during start-up of a nuclear reactor

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ABSTRACT

An analytical solution of the fractional model of the stiff point kinetics equations with multi-group of delayed neutrons is presented during a start-up of a nuclear reactor. The reactor start-up is a process of reactivity insertion which is mainly carried out by lifting control rods discontinuously. The inserting reactivity is a linear through the reactor start-up time and a constant after that. The analytical solution is based on the prompt jump approximation during a reactor start-up time and Laplace transform method otherwise that time. The response of neutron density is calculated for the ordinary and fractional models of the point kinetics equations with six groups of delayed neutrons. The results of neutron density for different fractional orders are performed and compared with the results of ordinary model. The comparison shows that the proposed analytical solution agrees well with the Zhang analytical method for the ordinary model with average one group of delayed neutrons. This agreement is decreasing in the case of six groups of delayed neutrons. In addition, the relations of the neutron density with the speed of lifting control rods and the fractional order are discussed.

1. Introduction

The new or reloaded reactor start-up requires special careful attention to problem of the control rods. The motion of the control rods into the nuclear reactor is related to reactor reactivity which is one of the important parameters. Inserting reactivity is mainly carried out by lifting control rods. The control rods are usually withdrawn in a discontinuous way during the start-up process of reactor. This process is equivalent to inserting reactivity linearly in a short period of time. It is very important to find out the relation between the speed of lifting control rods and the neutron density. In addition, When a pressurized-water reactor starts up in the cold state, temperature feedback is very small because of its lower system temperature and as a result the measurement system will not be sensitive enough. For this reason, Zhang et al. (2008) and Palma et al. (2009) developed analytical solutions of the point kinetics equations for the calculation of neutron density response during the cold start-up. Zhang et al. (2009) presented the dynamic simulation of cold start-up of a small pressurized-water reactor based on two-group point reactor model. Li et al. (2010) presented a new formula of neutron multiplication during start-up of pressurized-water reactor. Polo-Labarrios and Espinosa-Paredes (2012a,b) presented a numerical analysis of start-up pressurized-water reactor (PWR) with fractional neutron point kinetics equations. Seidi et al. (2014) generalized the analytical solution of neutron point kinetics equations with time-dependent external source. In addition, during delayed supercritical process and temperature feedback, the analytical approximation solutions of point reactor kinetics equations with average one group of delayed neutrons were presented by Chen et al. (2006, 2007, 2013), Li et al. (2007), Li et al. (2010), Nahla (2008) and Nahla and Zayed (2010). The better basis function (Li et al., 2009) and explicit appropriate basis function (Chen et al., 2015) were used to solve the stiff point kinetics equations with multi-group of delayed neutrons.

The ordinary point reactor kinetics equations with average one group of delayed neutrons take the following form Duderstadt and Hamilton (1976) and Stacey (2007)

$$\frac{dN(t)}{dt} = \left(\frac{\rho(t)-\beta}{l}\right)N(t) + \lambda C(t) + q, \tag{1}$$

$$\frac{dC(t)}{dt} = \frac{\beta}{l}N(t) - \lambda C(t) \tag{2}$$

$$\rho(t) = \begin{cases} \rho_s + rt, & 0 \leqslant t < t_0; \\ \rho_s + rt_0, & t \geqslant t_0. \end{cases} \tag{3}$$

where, $N(t)$ is the neutron density, t is the time, $\rho(t)$ is the reactivity, $C(t)$ is the precursor concentrations of average one group of

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delayed neutrons, $\beta=\sum_{i=1}^{I}\beta_i$ is the total fraction of delayed neutrons, β_i is the fraction of i-group of delayed neutrons, $\lambda=\frac{\beta}{\sum_{i=1}^{I}\beta_i/\lambda_i}$ is the decay constant of average one group of delayed neutrons, λ_i is the decay constant of i-group of delayed neutrons, l is the prompt neutron generation time, I is the total number of delayed neutron groups, q is the external neutron source, ρ_s is the sub-critical reactivity, r is the speed of lifting control rods and t_0 is the duration of each lifting control rods.

Zhang et al. (2008) used the prompt jump approximation and neglected the term $l\frac{d^2N(t)}{dt^2}$. Using these assumptions, the analytical solution of the ordinary point reactor kinetics equations with average one group of delayed neutrons were presented as follows:

$$N(t)=\begin{cases}\frac{\beta N_0+l\ q}{\beta-\rho_s-rt}, & 0\leqslant t<t_0;\\ \frac{\beta N_0+l\ q}{\beta-\rho_s-rt_0}\exp\left(\frac{\lambda(\rho_s+rt_0)}{\beta-\rho_s-rt_0}(t-t_0)\right) & \\ -\frac{l\ q}{\rho_s+rt_0}\left[1-\exp\left(\frac{\lambda(\rho_s+rt_0)}{\beta-\rho_s-rt_0}(t-t_0)\right)\right], & t\geqslant t_0.\end{cases}$$

In this work, we aim to present an analytical solution of the fractional point kinetics equations with multi-group of delayed neutrons during the start-up of a nuclear reactor, which is representing the generalization of the above stiff model. This fractional model with a fractional order $\propto$ can be written as (Ray and Patra, 2014)

$$D_t^\alpha N(t)=\left(\frac{\rho(t)-\beta}{l}\right)N(t)+\sum_{i=1}^{I}\lambda_i C_i(t)+q, \tag{4}$$

$$D_t^\alpha C_i(t)=\frac{\beta_i}{l}N(t)-\lambda_i C_i(t),\quad i=1,2,\ldots,I. \tag{5}$$

The analytical solution of this fractional model during a start-up process of a pressurized-water reactor is based on the prompt jump approximation, Laplace transform, the eigenvalues and corresponding eigenvectors of the coefficient matrix of this system.

This work is organized as follows: The basic definitions of the fractional calculus are introduced in Section 2. The analytical solutions of the matrix form of fractional point kinetics equations during and after the withdraw of the control rods are presented in Section 3. The numerical results of the analytical method are discussed in Section 4. The conclusion with a brief summary of the main findings is reported in Section 5.

2. Fractional calculus

The fractional calculus involves different definitions of the fractional operator as well as the Grünwald-Letnikov derivative, Riemann–Liouville integral and Caputo derivative. Let us introduce the definitions of the fractional operators as:

Definition 1 (*Podlubny, 1999; Ray and Patra, 2014*). The Grünwald-Letnikov fractional derivative of order α is defined as follows

$${}^{GL}D_t^\alpha f(t)=\lim_{h\to 0}\sum_{j=0}^{n}(-1)^j\binom{\alpha}{j}f(t-jh) \tag{6}$$

where α is the order of the derivative, h is the time step size, and $n=\frac{t}{h}$ is the number of time intervals.

Lubich proposed a second order numerical method for the fractional derivative of order α

$${}^{GL}D_t^\alpha f(t)\approx h^{-\alpha}\sum_{j=0}^{n}\Omega_j^\alpha f(t-jh) \tag{7}$$

where, $\Omega_0^\kappa=1$ and $\Omega_j^\alpha=\left(1-\frac{\alpha+1}{j}\right)\Omega_{j-1}^\alpha,\ j=1,2,\ldots$.

Definition 2 (*Podlubny, 1999; Ray and Patra, 2014*). The Riemann–Liouville fractional integral operator of order $\alpha\geqslant 0$ of a function $f(t)$ is defined as

$$I_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(\xi)}{(t-\xi)^{\alpha-1}}d\xi,\quad \alpha>0 \tag{8}$$

where, $\Gamma(\alpha)$ is the gamma function whose argument is (α).

According to the Riemann–Liouville definition, we get

$$I_t^\alpha t^m=\frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)}t^{m+\alpha} \tag{9}$$

Definition 3 (Caputo, 1969). The fractional derivative of function $f(t)$ in the Caputo sense is defined as

$$\begin{aligned}D_t^\alpha f(t)\ &=\tfrac{1}{\Gamma(m-\alpha)}\int_0^t(t-\xi)^{m-\alpha-1}f^{(m)}(\xi)d\xi,\\ &m-1<\alpha<m,\ m\in N,\ t>0.\end{aligned} \tag{10}$$

where $f^{(m)}(\xi)=\frac{\partial^m f(\xi)}{\partial\xi^m}$

Based on the Caputo definition, the following result is obtained

$$D_t^\alpha t^m=\begin{cases}\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}t^{m-\alpha}, & m\in N;\\ 0, & m=0.\end{cases} \tag{11}$$

For the Riemann–Liouvill fractional integral and Caputo fractional derivative (Podlubny, 1999), we get

$$I_t^\alpha D_t^\alpha f(t)=f(t)-\sum_{r=0}^{m-1}f^{(r)}(0_+)\frac{t^r}{r!},\quad m-1<\alpha\leqslant m. \tag{12}$$

3. Analytical solution of the fractional model

To reach the analytical solution of the stiff fractional point kinetics equations, we divide the problem time to two intervals: through the control rods withdraw $0\leqslant t<t_0$ and after that $t\geqslant t_0$. The time of lifting control rods t_0 is short which is the average of half life time for the precursor delayed neutrons i.e. $t_0\leqslant\frac{\ln 2}{\lambda}$. This means that, during lifting control rods, the precursor concentration of delayed neutrons are constants and equal its initial values (Zhang et al., 2008)

$$C_i(t)=C_i(0)=\frac{N_0\beta_i}{l\lambda_i},\quad 0\leqslant t<t_0 \tag{13}$$

Substituting Eq. (13) into Eq. (4), gives

$$D_t^\alpha N(t)=\frac{\rho_s+r\ t-\beta}{l}N(t)+\frac{N_0\beta+l\ q}{l},\quad 0\leqslant t<t_0 \tag{14}$$

Multiplying both sides by l, yields

$$l\ D_t^\alpha N(t)=(\rho_s+r\ t-\beta)N(t)+N_0\beta+l\ q,\quad 0\leqslant t<t_0 \tag{15}$$

Using the prompt jump approximation, left hand side is very small comparing with the right hand side i.e. $l\ D_t^\alpha N(t)\to 0$, then we have

$$N(t)=\frac{N_0\beta+l\ q}{\beta-\rho_s-r\ t},\quad 0\leqslant t<t_0 \tag{16}$$

After lifting control rods, the reactivity becomes a constant and the Eqs. (4) and (5) can be rewritten in matrix form as

$$D_t^\alpha|\Psi(t)\rangle=\mathbf{A}|\Psi(t)\rangle+|\mathbf{Q}\rangle,\quad t\geqslant t_0. \tag{17}$$

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