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Solution of the two dimensional diffusion and transport equations in a rectangular lattice with an elliptical fuel element using Fourier transform methods: One and two group cases

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ABSTRACT

A method for solving the diffusion and transport equations in a rectangular lattice cell with an elliptical fuel element has been developed using a Fourier expansion of the neutron flux. The method is applied to a one group model with a source in the moderator. The cell flux is obtained and also the associated disadvantage factor. In addition to the one speed case, we also consider the two group equations in the cell which now become an eigenvalue problem for the lattice multiplication factor. The method of solution relies upon an efficient procedure to solve a large set of simultaneous linear equations and for this we use the IMSL library routines. Our method is compared with the results from a finite element code. The main drawback of the problem arises from the very large number of terms required in the Fourier series which taxes the storage and speed of the computer. Nevertheless, useful solutions are obtained in geometries that would normally require the use of finite element or analogous methods, for this reason the Fourier method is useful for comparison with that type of numerical approach. Extension of the method to more intricate fuel shapes, such as stars and cruciforms as well as superpositions of these, is possible.

1. Introduction

The earliest known application of Fourier series to lattice problems is that of Feynman and Welton (1946) who used it to obtain the criticality conditions of a mixture of moderator and fissionable material in the form of a lattice. Conditions for replacing the lattice with effective cross sections were also investigated by Feynman and Welton. The basic assumption of the method was that the mean free path was the same throughout the system although both absorption and scattering cross section could vary. The work was based upon the integral form of the transport equation. It was this early paper that inspired the present work and our first contribution was made by Williams and Wood (1972) to streaming and lattice problems. In 1967 Zweifel and Mannan (1967) used the Fourier expansion to solve a two-dimensional lattice problem using diffusion theory and we shall comment on that below. The renewed interest in the Fourier method stems from the much more powerful methods now available for summing the very slowly converging series that arise. This is not to say that we have solved the convergence problem and we offer this method not as a benchmark method, but as a very useful scoping tool that can evaluate the effect of geometrically complicated cell structure on integral reactor parameters.

We will show why, in general, it is not practical to use the Fourier method in transport theory when the total cross section varies with position. However, this cross section restriction in transport theory still enables the errors in diffusion theory to be assessed for special cases and also allows some useful non-trivial transport problems to be solved, as we will show in our section on numerical results. The method is very useful indeed for solving a variety of practical diffusion theory problems.

In addition to the one speed case, we also consider the two group equations in the cell which now becomes an eigenvalue problem for the lattice multiplication factor. This too is evaluated for diffusion and transport theories, and fast and thermal disadvantage factors are obtained. We are also able to assess the accuracy of the spatially constant source approximation used in the one group method and it is found that there is around a 7% variation of this slowing down source across the moderator region.

Comparisons with the Wigner–Seitz method is made for both one and two group methods and the associated error calculated. A curious phenomenon in which the disadvantage factor is seen to go through a minimum as the cell size increases for a given fuel

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rod size is confirmed for the rectangular cell and is shown to be absent from the Wigner–Seitz approximation. This shows that this is a geometric matter and is not connected with the use of diffusion or transport theory. The rate of convergence of the P_N method is shown by approximation of the transport relation with a continued fraction expansion, which also shows the relationship with the SP_N method.

2. Diffusion theory

In order to illustrate the problem, let us first deal with the diffusion equation with spatially variable cross sections in an infinite lattice array. Consider then the one speed, two-dimensional diffusion equation for the neutron flux $\phi(x, y)$, viz:

$$\frac{\partial}{\partial x}D(x,y)\frac{\partial}{\partial x}\phi(x,y)+\frac{\partial}{\partial y}D(x,y)\frac{\partial}{\partial y}\phi(x,y)-\Sigma_a(x,y)\phi(x,y)+Q(x,y)=0 \quad (1)$$

We will solve this equation in a periodic lattice in which the period in x is T_1 and that in y is T_2. Within the cell, the diffusion coefficient $D(x, y)$, the absorption cross section $\Sigma_a(x, y)$ and the source $Q(x, y)$ may vary in an arbitrary fashion, although generally there will be distinct regions in the cell, each with different but constant properties. In view of the periodicity of the problem we may seek solutions in the form

$$\phi(x,y)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\phi_{nm}e^{i\alpha_n x+i\beta_m y} \quad (2)$$

where

$$\alpha_n=\frac{2\pi n}{T_1} \quad \text{and} \quad \beta_m=\frac{2\pi m}{T_2} \quad (3)$$

This solution ensures that the gradient of the flux along the boundary of the cell is zero. Both $D(x, y)$, $\Sigma_a(x, y)$ and $Q(x, y)$ are also periodic and of course known. They can be expanded therefore as

$$\begin{aligned}D(x,y)&=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}D_{nm}e^{i\alpha_n x+i\beta_m y}\\ \Sigma_a(x,y)&=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\Sigma_{a,nm}e^{i\alpha_n x+i\beta_m y}\\ Q(x,y)&=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}Q_{nm}e^{i\alpha_n x+i\beta_m y}\end{aligned} \quad (4)$$

We will discuss a method for calculating D_{nm}, $\Sigma_{a,nm}$ and Q_{nm} below. Inserting Eqs. (2) and (4) into (1), multiplying by $e^{-i\alpha_k x-i\beta_\ell y}$, integrating over the cell area and using

$$\int_{-T_1/2}^{T_1/2}dx\int_{-T_2/2}^{T_2/2}dy e^{i(\alpha_n-\alpha_k)x+i(\beta_m-\beta_\ell)y}=T_1T_2\delta_{nk}\delta_{m\ell} \quad (5)$$

leads to

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\phi_{nm}\{[\alpha_n(\alpha_n+\alpha_{r-n})+\beta_m(\beta_m+\beta_{s-m})]D_{r-n,s-m}+\Sigma_{a,r-n,s-m}\}=Q_{rs} \quad (6)$$

where $-\infty < r < \infty$, $-\infty < s < \infty$. Eq. (6) constitutes a set of linear equations for the expansion coefficients ϕ_{nm}.

Let us now discuss how to calculate the expansion coefficients in Eqs. (4). The internal structure of the cell will be assumed symmetric about the centre and for generality we will assume that the fuel element is elliptical in cross section. It is possible to have several concentric ellipses but we settle on a single one. We assume that the major axis is in the x direction and has radius a and the minor axis is in the y direction with radius b. Thus the equation of the perimeter is

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1 \quad (7)$$

For $a > b$, the eccentricity is defined as $\varepsilon=\sqrt{a^2-b^2}/a$. Now in order to obtain the expansion coefficients in (4) let us consider a general function

$$f(x,y)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}f_{nm}e^{i\alpha_n x+i\beta_m y} \quad (8)$$

This may be written from symmetry as

$$\begin{aligned}f(x,y)=f_{00}+2\sum_{n=1}^{\infty}f_{n0}\cos(\alpha_n x)+2\sum_{n=1}^{\infty}f_{0n}\cos(\beta_n y)+4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}f_{nm}\\ \times\cos(\alpha_n x)\cos(\beta_m y)\end{aligned} \quad (9)$$

This of course is also the form of the flux $\phi(x, y)$. Using (5) we also have

$$f_{k\ell}=\frac{1}{T_1T_2}\int_{-T_1/2}^{T_1/2}dx\int_{-T_2/2}^{T_2/2}dy f(x,y)e^{-i\alpha_k x-i\beta_\ell y} \quad (10)$$

Now the function $f(x, y)$ is such that, with $A = A_0 + A_1$, where $A = T_1T_2$ and $A_1 = \pi ab$,

$$\begin{aligned}f(x,y)&=f_1 \quad \text{inside the ellipse}\\ f(x,y)&=f_0 \quad \text{outside the ellipse}\end{aligned} \quad (11)$$

Thus

$$f_{k\ell}=\frac{1}{T_1T_2}\left[\int\int_{A_1}dxdy f_1 e^{-i\alpha_k x-i\beta_\ell y}+\int\int_{A_0}dxdy f_0 e^{-i\alpha_k x-i\beta_\ell y}\right] \quad (12)$$

But we may re-arrange this as

$$\begin{aligned}f_{k\ell}&=\frac{1}{T_1T_2}\left[\int\int_{A}dxdy f_0 e^{-i\alpha_k x-i\beta_\ell y}+(f_1-f_0)\int\int_{A_1}dxdy e^{-i\alpha_k x-i\beta_\ell y}\right]\\ &=f_0\delta_{k0}\delta_{\ell 0}+(f_1-f_0)\frac{1}{T_1T_2}\int\int_{A_1}dxdy e^{-i\alpha_k x-i\beta_\ell y}\end{aligned} \quad (13)$$

From the symmetry of $f(x, y) = f(-x, -y) = f(-x, y) = f(x, -y)$, we may write the integral in (13) as

$$\frac{1}{T_1T_2}\int\int_{A_1}dxdy e^{-i\alpha_k x-i\beta_\ell y}=\frac{4}{T_1T_2}\int_0^a dx\int_0^{b\sqrt{a^2-x^2}/a}dy\cos(\alpha_k x)\cos(\beta_\ell y) \quad (14)$$

which reduces to (Gradshteyn and Ryzhik, 1994)

$$\frac{ab}{T_1T_2\sqrt{\left(\frac{ak}{T_1}\right)^2+\left(\frac{b\ell}{T_2}\right)^2}}J_1\left(2\pi\sqrt{\left(\frac{ak}{T_1}\right)^2+\left(\frac{b\ell}{T_2}\right)^2}\right) \quad (15)$$

Thus from (13)

$$f_{00}=f_0+(f_1-f_0)\frac{\pi ab}{T_1T_2} \quad (16)$$

and for all other values of k and ℓ,

$$f_{k\ell}=(f_1-f_0)\frac{ab}{T_1T_2\sqrt{\left(\frac{ak}{T_1}\right)^2+\left(\frac{b\ell}{T_2}\right)^2}}J_1\left(2\pi\sqrt{\left(\frac{ak}{T_1}\right)^2+\left(\frac{b\ell}{T_2}\right)^2}\right) \quad (17)$$

In the special case of a circle in a square, we have

$$f_{00}=f_0+(f_1-f_0)\frac{\pi a^2}{T^2} \quad (18)$$

and