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Stochastic response of fractional-order van der Pol oscillator

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Abstract We studied the response of fractional-order van de Pol oscillator to Gaussian white noise excitation in this letter. An equivalent integral-order nonlinear stochastic system is obtained to replace the given system based on the principle of minimum mean-square error. Through stochastic averaging, an averaged Itô equation is deduced. We obtained the Fokker–Planck–Kolmogorov equation connected to the averaged Itô equation and solved it to yield the approximate stationary response of the system. The analytical solution is confirmed by using Monte Carlo simulation.

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Keywords fractional-order van de Pol oscillator, Gaussian white noise, stationary response, equivalent nonlinear system method, stochastic averaging

As a generalization of the classical calculus, fractional calculus has been applied to various fields in the past decades. The major advantage of the fractional calculus originates in the fact that it has been proven to be an excellent tool to describe the memory and hereditary properties of various materials and processes. Also, many practical systems (e.g., electromagnetic waves in dielectrics)[1] can be described more adequately through the fractional-order differential equations. On the other hand, the van de Pol (VDP) oscillator, proposed originally as a model of vacuum tube circuits and later applied widely to various fields was one of the most famous self-excited systems. Recently, the model of classical VDP oscillator has been further developed, and the fractional-order VDP oscillator has been formulated[2] and studied by means of different numerical methods.[3,4] In this letter, we investigate the stationary response of fractional-order VDP oscillator to external Gaussian white noise excitation. The critical procedure is to obtain an equivalent integral-order stochastic system and use the stochastic averaging technique to study the equivalent integral-order stochastic system.

Consider a fractional-order VDP oscillator externally excited by a Gaussian white noise. The motion equation has the following form

$$D^{\alpha}X(t)+(-\beta_0+\beta_1 X(t)^2)\dot{X}(t)+\omega_0^2 X(t)=\xi(t), \tag{1}$$

where ω_0, β_0, and β_1 are positive parameters. $D^{\alpha}X(t)$ is the fractional derivative under the defi-

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nition of Riemann–Liouville, and it reads as

$$D^{\alpha}X(t) = \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^t \frac{X(t-\tau)}{\tau^{\alpha-1}}\mathrm{d}\tau, \ \ 1 < \alpha \leqslant 2, \tag{2}$$

where $\Gamma(\cdot)$ is Gamma function and α is the fractional order. $\xi(t)$ is Gaussian white noise having intensity $2d$. β_0, β_1, d are of the same order of ε, here ε is a small positive parameter.

Since the fractional order satisfies $1 < \alpha \leqslant 2$, the fractional derivative term has contributions to both inertia and damping. Hence, introduce the following equivalent system

$$m(A)\ddot{X} + [-\beta_0 + \beta(A) + \beta_1 X^2]\dot{X} + \omega_0^2 X = \xi(t), \tag{3}$$

where $m(A)$ and $\beta(A)$ are the coefficients of equivalent inertia and damping forces, respectively, and $X = X(t)$.

The error between Eqs. (1) and (3) is

$$e = m(A)\ddot{X} - D^{\alpha}X + \beta(A)\dot{X}. \tag{4}$$

The necessary conditions for minimum mean square error are

$$\partial E[e^2]/\partial m = 0, \ \ \partial E[e^2]/\partial \beta = 0. \tag{5}$$

Substituting Eq. (4) into Eq. (5) yields

$$\begin{aligned} E[\ddot{X}^2 m(A) - \ddot{X}D^{\alpha}X + \beta(A)\ddot{X}\dot{X}] &= \lim_{T\to\infty}\frac{1}{T}\int_0^T (\ddot{X}^2 m(A) - \ddot{X}D^{\alpha}X + \beta(A)\ddot{X}\dot{X})\mathrm{d}t = 0,\\ E[\ddot{X}\dot{X} m(A) - \dot{X}D^{\alpha}X + \beta(A)\dot{X}^2] &= \lim_{T\to\infty}\frac{1}{T}\int_0^T (\ddot{X}\dot{X} m(A) - \dot{X}D^{\alpha}X + \beta(A)\dot{X}^2)\mathrm{d}t = 0. \end{aligned} \tag{6}$$

Assume that the solution of Eq. (3) is of the form[5]

$$\dot{X}(t) = -A(t)\omega(A)\sin\Theta(t), \ \ \ X(t) = A(t)\cos\Theta(t), \tag{7}$$

where

$$\Theta(t) = \Gamma(t) + \omega(A)t, \ \ \omega(A) = \omega_0/\sqrt{m(A)}, \tag{8}$$

with $\omega(A)$ being frequency of oscillator. Differentiating Eq. (7) with respect to t leads to

$$\ddot{X}(t) = -\sin\Theta\frac{\mathrm{d}}{\mathrm{d}t}A\omega(A) - A\omega(A)\left(\omega(A) + \frac{\mathrm{d}\Gamma}{\mathrm{d}t}\right)\cos\Theta. \tag{9}$$

Since $\mathrm{d}A(t)/\mathrm{d}t$ and $\mathrm{d}\Gamma(t)/\mathrm{d}t$ are approximate to 0 and $\Theta(t-\tau)$ is close to $\Theta(t) - \tau\omega(A)$ while τ