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Letter

Direct and noisy transitions in a model shear flow

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ABSTRACT

The transition to turbulence in flows where the laminar profile is linearly stable requires perturbations of finite amplitude. "Optimal" perturbations are distinguished as extrema of certain functionals, and different functionals give different optima. We here discuss the phase space structure of a 2D simplified model of the transition to turbulence and discuss optimal perturbations with respect to three criteria: energy of the initial condition, energy dissipation of the initial condition, and amplitude of noise in a stochastic transition. We find that the states triggering the transition are different in the three cases, but show the same scaling with Reynolds number.

1. Introduction

In parallel shear flows like pipe flow, plane Couette flow or Poiseuille flow and in boundary layers like the asymptotic suction boundary layer or the Blasius profile, turbulence appears when the laminar profile is linearly stable against perturbations [1]. Accordingly, finite amplitude perturbations are required to trigger turbulence, a scenario referred to as by-pass transition [2]. Many studies in the above flows have shown that the transition to turbulence is associated with the presence of 3D exact coherent states [3]. They appear in saddle-node bifurcations which in the state space of the system create regions of initial conditions that do not decay to the laminar profile, but instead are attracted towards the node-state [4]. As the Reynolds number increases, the region widens, the node state undergoes further bifurcations and chaotic attractors or saddles are formed [5–7]. Initial conditions can only trigger turbulence when they reach into that interior region, i.e., cross the stable manifold of the saddle state on the boundary of the region [8]. An "optimal" perturbation is one that can trigger turbulence and at the same time is a minimum of a prescribed functional. Popular is an optimization based on amplification or energy gain over a given interval in time [9–17] or on the total time-averaged dissipation [18,19]. Because they take the time-evolution into account, they connect to optimization problems in control theory [20,21].

We simplify matters here and focus on a geometric optimization by identifying initial conditions that will eventually become turbulent, without regard of the time it takes for them to become turbulent. The states are optimized so that a certain quadratic function, such as energy content or dissipation, is extremal: it is a maximum in the sense that all initial conditions with a lower value of the quadratic function will not become turbulent, and it is minimal in that the first initial conditions that become turbulent have values larger than this optimum. At the optimal value there will then be at least one trajectory which neither becomes turbulent nor returns to the laminar profile: it lies on the stable manifold of the edge state [8], so that the optimum is reached when the isocontours of the optimization functional touch the stable manifold of the edge state (similar descriptions of the state space structure can be found in Refs. [17,19,22,23]).

2. The Model

To fix ideas and to keep the mathematics as simple as possible, we take the 2D model introduced by Baggett and Trefethen [24]. The model we use is one of a set of many low-dimensional models of various levels of complexity [20,22,25–31]. It has a non-normal linear part and an energy conserving nonlinearity, and, this being the most important feature for the present application, it is 2D so that the entire phase space can be visualized (a property it shares with the illustrative model of Ref. [20]). Despite its simplicity, the model can be used to illustrate several features of the transition mechanisms in shear flows.

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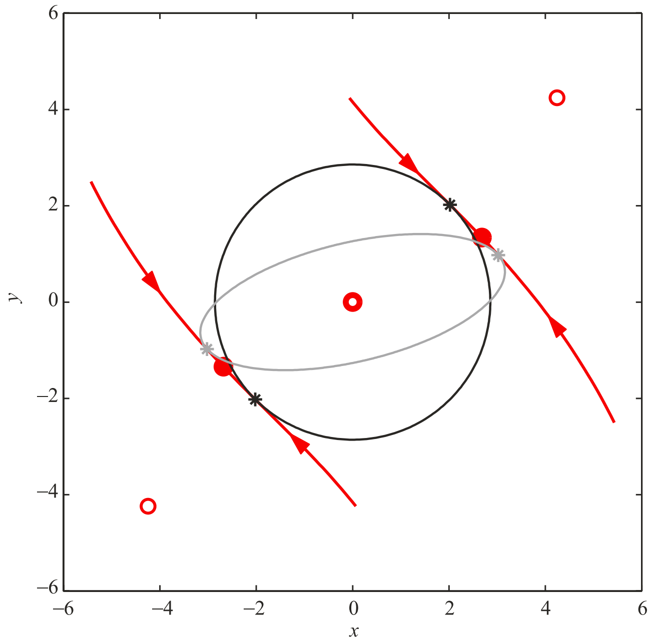

Fig. 1. (Color online) State space of the 2D model for the transition to turbulence for $R = 3$. The open symbols mark the stable fixed point at the origin ("laminar" state) and the two nodes from the bifurcation ("turbulent" states). The full symbols are the edge states, and the red lines indicate the stable manifolds of the edge states. The black circle and the gray ellipsoid indicate the states where the energy (7) and the noise functional (12) are minimal, respectively. The points where they touch the stable manifolds are indicated by stars.

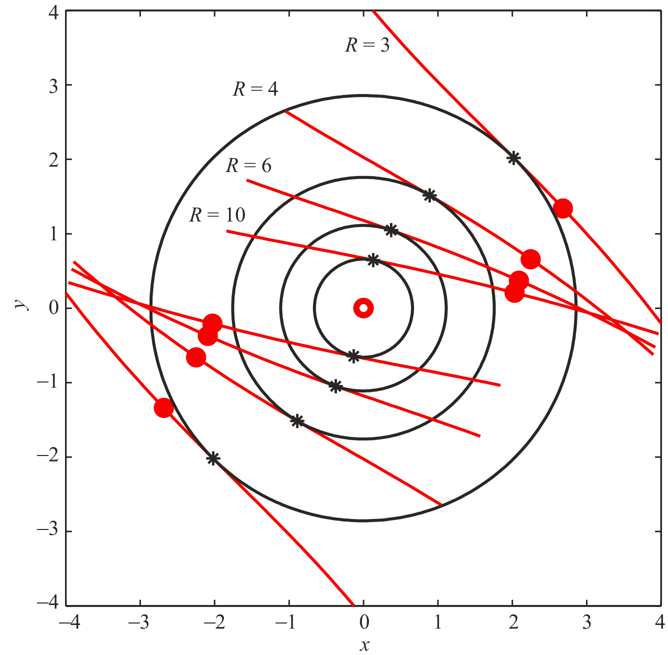

Fig. 2. (Color online) Optimal states in energy for different R. The open symbol in the middle is the laminar state. States of fixed energy are indicated by circles, and the points where they touch the stable manifolds (red lines) of the edge states (indicated as full symbols) are the points marked by stars. One notes that as R increases, the manifolds become more parallel to the x-axis, and the point of contact approaches the origin from the y-axis.

The model has two variables, which may be thought of as measuring the amplitudes of streaks x and vortices y (see also Ref. [32]), and one parameter R that plays the role of the Reynolds number

$$\dot{x} = -x/R + y - y\sqrt{x^2 + y^2}, \tag{1}$$

$$\dot{y} = -2y/R + x\sqrt{x^2 + y^2}. \tag{2}$$

In order to highlight more clearly what happens near the origin, we magnify by rescaling the variables with the Reynolds number R (see Ref. [33]), i.e., we redefine the amplitudes $x = x'/R^2, y = y'/R^2$ and the time $t = Rt'$ such that (with the primes dropped)

$$\dot{x} = -x + Ry - y\sqrt{x^2 + y^2}/R, \tag{3}$$

$$\dot{y} = -2y + x\sqrt{x^2 + y^2}/R. \tag{4}$$

Time evolution under the nonlinear terms alone preserves $x^2 + y^2$, which may be thought of as a kind of energy, so that the nonlinear terms are "energy" conserving. For $R < R_c = \sqrt{8}$ the only fixed point is $x = y = 0$, henceforth referred to as the "laminar" fixed point. At $R = R_c$, symmetry related fixed points appear at (x_c, y_c) and $(-x_c, -y_c)$, with

$$x_c = R(2R \pm 2\sqrt{R^2 - 8})/D_\pm, \tag{5}$$

$$y_c = R(R^2 - 4 \pm R\sqrt{R^2 - 8})/D_\pm, \tag{6}$$

where $D_\pm = \sqrt{8 + 2R^2 \pm 2R\sqrt{R^2 - 8}}$. The two fixed points closest to the origin are unstable, hence are saddle states, and the two further out are stable and hence nodes. The saddle states are the "edge states" [8] and the node states are in the regions where turbulence would form, if more degrees of freedom were available. Nevertheless, we will refer to them as the "turbulent" states.

For $R \to \infty$, the saddles are to leading order in $1/R$ located at $\pm(2, 2/R)$, which in the original coordinates represents an approach to the origin like $\pm(2/R^2, 2/R^3)$. The stable manifolds rotate so as to become parallel to the x-axis, as we will see in the following.

3. Optimal Initial Conditions of Minimal Energy

The Euclidean distance to the origin can be obtained from a quadratic form

$$E = (1/2)(x^2 + y^2), \tag{7}$$

which has the form of kinetic energy. This assignment is further supported by the observation that E is preserved under time evolution by the nonlinear terms alone. In the sense described in the introduction, optimality with respect to this energy functional thus means the largest value up to which all trajectories return to the laminar state, and the smallest one where the first trajectories that evolve towards the turbulent state become possible. On the boundary between these two cases are states that neither return to laminar nor become turbulent, that lie on the stable manifold of the edge state. Geometrically, we are thus looking for the circle with the largest radius that we can draw around the origin that just touches the stable manifold. Algorithmically, we find this point by a modified edge tracking which minimizes the energy (7) as described in the Appendix.

An example of such an optimal circle is given in Fig. 1, and its variation with R is shown in Fig. 2. As the Reynolds number increases, the fixed point moves towards $(2, 0)$ on the abscissa, and the stable manifold rotates to being parallel to the abscissa. The point of contact between circles of equal energy and the stable manifold moves away from the edge state, approaches the y-axis and moves inwards to the origin like $1/R$.

In an insightful discussion of the energy functional, Cossu [23] notes that in the time-derivative of the energy functional only the linear parts of the equations of motion remain and that the nonlinear ones drop out because energy is preserved. This observation allows to define a necessary condition for the location