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Letter

Near continuum boundary layer flows at a flat plate

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ABSTRACT

The problem of boundary layer flows at a flat plate surface with velocity-slip and temperature-jump boundary conditions is analyzed. With the velocity slip conditions, there are multiple physical factors lumped together, and the boundary layer solutions significantly change their behaviors. The self-similarity in the solutions degenerates, however, the problem is still an ordinary differential equation which can be solved. Shooting methods are applied to solve the flowfield. The results include velocity and temperature for both the surface and flowfield. Unlike the traditional Blasius flat plate boundary layer solutions which are self-similar through all the plate boundary layer, the new solutions indicate that the front tip is actually a singularity point, especially at locations within one mean free path from the leading edge.

The problem of incompressible boundary layer at a flat plate with non-slip and constant heat flux/temperature was investigated successfully. Blasius et al. [1,2] introduced a coordinate transformation method, and the governing Navier–Stokes partial differential equations (NSEs) for incompressible flows were transformed into a single ordinary differential equation, from which a universal velocity profile can be obtained for the whole flowfield. In addition, surface properties, such as the friction coefficients, are obtained theoretically. In the literature, there are many numerical and experimental studies as well. The solutions for boundary layer along a flat plate were derived and they can find many applications, e.g., crude estimations for drags over an airfoil.

As technologies and sciences advance, many new applications involving boundary layers emerge, and rarefication effects must be considered. For these flows, the traditional NSEs are not directly applicable. The rarefication effects are described by the Knudsen (Kn) number [3]

$$Kn = \lambda/L, \tag{1}$$

where λ is the molecule mean free path (MFP), and L the characteristic length. Larger Kn number flows can be created by large MFP (e.g., in space engineering), or small characteristic lengths, e.g., shock waves, gas flows inside micro-electro-mechanical systems/nano-electro-mechanical system (MEMS/NEMS). For example, Tretheway and Meinhart [4] reported in a micro-channel, with a very thin coating, the velocity slip can be quite apparent. Within the continuum flow regime ($Kn < 0.001$) with a small MFP, the NSEs apply well with the non-slip velocity boundary conditions. As the Kn number continues to increase, flows change to the velocity-slip and temperature-jump ($0.001 < Kn < 0.01$) regime. With further larger MFPs, flows can be transitional ($0.01 < Kn < 10$) and free molecular ($10 < Kn$). Within the continuum flow regime, Blasius's solutions are well developed; within the free molecular flow regime, the surface and flowfield solutions were obtained by Schaaf and Chambre [3] and Cai [5]. Within the transitional flow regime, we rely on numerical simulation methods. With the velocity slip and temperature jump regime, there has been some progress [6,7], and the major goal of this paper aims to continue the discussions on flows within this regime.

The Blasius boundary layer on a flat plate with non-slip boundary conditions It is well known that for an incompressible gas flow over a flat plate, the NSEs can be simplified as

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\mathrm{d}U}{\mathrm{d}x} + \nu\frac{\partial^2 u}{\partial y^2}, \\ &u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_\mathrm{p}}\frac{\partial^2 T}{\partial y^2}. \end{aligned} \tag{2}$$

The plate surface conditions are listed as

$$\begin{aligned} &u(x, 0) = v(x, 0) = 0, \qquad u(x, +\infty) = U, \\ &T(x, 0) = T_\mathrm{w}, \qquad T(x, \infty) = T_\mathrm{e}. \end{aligned} \tag{3}$$

The exact solutions to boundary layer flows over a flat plate were developed by Blasius, and were explained more conveniently by White [8]. A stream function $\psi(x, \eta)$ can be adopted via a variable

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transformation from (x, y) to (x, η)

$$\begin{aligned} &\psi = \sqrt{2\nu U x} f(\eta), \qquad \eta = y\sqrt{U/(2\nu x)} = y/\delta(x), \\ &\delta(x) = \sqrt{2\nu x/U}, \qquad u = \frac{\partial \psi}{\partial y} = Uf'(\eta), \\ &v = -\frac{\partial \psi}{\partial x} = (\eta f' - f)\sqrt{\frac{\nu U}{2x}}, \end{aligned} \tag{4}$$

where ν is the kinetic viscosity, and U the free stream velocity, f is a single variable function,

$$f'''(\eta) + f(\eta)f''(\eta) = 0. \tag{5}$$

And the boundary conditions are transformed into

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1. \tag{6}$$

The above new equation is a concise ordinary differential equation (ODE), not a partial differential equation (PDE). As a result, there are exact solutions, and the solving procedure is simple. By using the shooting method [8], the above two-point boundary value problem can be solved numerically. Some plate surface properties, such as friction coefficients, can be obtained analytically.

For the temperature field, with a transformation of variable [8]

$$\Theta(\eta) = \frac{T - T_e}{T_w - T_e}, \tag{7}$$

the governing equation and boundary conditions for temperature are

$$\Theta'' + Prf(\eta)\Theta' = 0, \tag{8}$$

$$\Theta(0) = 1, \qquad \Theta(\infty) = 0, \tag{9}$$

where Pr is the Prandtl number. The exact solution is

$$\Theta(\eta) = \int_\eta^\infty e^{-Pr\int_0^\eta f \mathrm{d}s} \mathrm{d}\eta / \left(\int_0^\infty e^{-Pr\int_0^\eta f \mathrm{d}s} \mathrm{d}\eta \right). \tag{10}$$

Slip velocity boundary conditions ($0.001 < Kn < 0.01$) The previous section is the foundation for the work in this paper on velocity-slip and temperature jump boundary conditions. There is much related work in the literature, and they are reviewed as follows.

The first category of work concentrated on explanations of the velocity-slip and temperature-jump boundary conditions. Maxwell was the first one (1890) who discovered that due to the existence of the Knudsen layer close to the surface, the boundary condition at the plate surface shall have discontinuity effects, the velocity and temperature boundary conditions shall be modified [9,10]. Very soon, Smoluchowski [11,12] published two papers reporting similar results but with a separate method. Payne [13] relaxed Smoluchowski's assumption, and provided more general results where a Maxwellian type boundary condition is merely a special scenario. By using the gaskinetic theory, and a multi-scale expansion method, Wu et al. [14] provided a slightly different, detailed explanation on the inner and outer solutions for flows in the velocity slip regime. It was emphasized by many researchers [15–17], that when surface curvatures exist, then extra terms shall also be included in the velocity-slip boundary condition. Such past work concentrated on derivations for these velocity-slip boundary condition, rather than applying these new boundary conditions to the similarity solutions for boundary layer flows. Higher order slip boundary conditions [16] were also proposed.

A comprehensive review on experiments and numerical simulations of rarefied gas flows over a flat plate is available in the literature [18]. The non-equilibrium effects on the leading edge of a flat plate is reported [19]. Those work did not follow the approach for similarity solutions by Blasius. Recently, Matthews and Hill [20,21] discussed their work on slip flows over a flat plate with more general slip boundary conditions. In their work, no variable transformation was introduced, and the work did not include temperature jump boundary conditions. Martin and Boyd [6] reported their work on similarity solutions for flows over a flat plate with velocity-slip and temperature-jump conditions. They introduced an extra parameter K_1 which is related to $x^{1/2}$, in addition to the two transformed variables (x, η).

The velocity-slip boundary conditions can be expressed as

$$\begin{aligned} &u_s(x,0) = u_g - u_w = \frac{2-\sigma_M}{\sigma_M}\lambda \left.\frac{\partial u}{\partial n}\right|_w + \frac{3}{4}\frac{\nu}{T_g}\left.\frac{\partial T}{\partial x}\right|_w, \\ &v(x,0) = 0, \qquad u(x,\infty) = U, \end{aligned} \tag{11}$$

where u_s is the wall slip velocity, i.e., the velocity difference between gas and the wall surface, u_g the gas bulk velocity adjacent to the wall, u_w the wall surface velocity, $\partial u/\partial n$ the gas velocity gradient normal to the wall, σ_M a tangential momentum accommodation coefficient, T_g the gas temperature, λ the MFP for a gas flow and can be described by the hard sphere model, $\lambda = m/(\sqrt{2}\pi d^2\rho)$ for a molecule of a diameter d, m the molecular mass, and ρ the gas density which is usually of an ordinary value for gas flows inside MEMS. In general, the term containing temperature gradient in Eq. (11) is negligible when compared with the velocity gradient term.

With the same coordinate transformations, Eq. (4), the new velocity boundary condition changes

$$\frac{\partial}{\partial n} = \sqrt{\frac{U}{2\nu x}}\frac{\partial}{\partial \eta}, \qquad f'(0) = \sqrt{\frac{\lambda U}{2\nu}}\frac{2-\sigma_M}{\sigma_M}\sqrt{\frac{\lambda}{x}} f''(0). \tag{12}$$

By using a crude gaskinetic estimation [22]

$$\nu = \frac{\lambda}{3}\sqrt{\frac{8RT}{\pi}}, \tag{13}$$

Eq. (12) can be transformed as

$$\frac{f'(0)}{f''(0)} = \sqrt{\frac{3}{4}\sqrt{\frac{\gamma\pi}{2}} M_0 \frac{\lambda}{x}} \frac{2-\sigma_M}{\sigma_M}, \qquad f'(\infty) = 1, \tag{14}$$

where the right hand side of the expression is defined as the slip coefficient, M_0 is the free-stream Mach number, and γ the specific heat ratio. Since several factors are combined together, it is evident that different changes of variables may achieve the same effect; for example, with a larger σ_M, or at a station x/λ further downstream from the leading edge. Boundary conditions, Eq. (14), involve a normalized factor of x/λ. Hence, at different stations, the boundary conditions vary. Equations (5) and (14) are compatible, with the transformed coordinate system of (x, η). The governing equation contains x implicitly through η. Martin & Boyd [6] introduced an extra parameter K_1 which involves the Kn_x and Reynolds Re_x number. The characteristic length for both is the distance from the leading edge, i.e., x. As can be derived, $K_1 \propto Kn_x Re_x^{1/2} \propto x^{-1/2}$, and it is improper to apply the derivative computation, due to the chain rule between the old coordinates (x, y) to the new coordinate system (x, η)

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} + \frac{\partial K_1}{\partial x}\frac{\partial}{\partial K_1}. \tag{15}$$

Kumaran and Pop [23] investigated one related isothermal flow problem with a moving plate. Different from introducing a new parameter K_1 as Martin's work, they performed a small parameter expansion method. Their approach is obviously improper because the slip coefficient which is defined in Eq. (14) includes a variable x which cannot be considered as a constant.

For gas flows inside MEMS/NEMS, or over a flat plate, the gas density is actually relatively constant; hence, the MFP does not