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Strength, stability and three dimensional structure of mean motion resonances in the solar system

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ABSTRACT

In the framework of the circular restricted three body problem we show that the numerically computed strength $SR(e, i, \omega)$ is a good indicator of the strength and width of the mean-motion resonances in the full space (e, i, ω). We present a survey of strengths in the space (e, i) for typical interior and exterior resonances. The resonance strength is highly dependent on (e, i, ω) except for exterior resonances of the type 1:k for which the dependence with (i, ω) is softer. Such resonances are thus strong even for retrograde orbits. All other resonances are weaker at very-high eccentricities for $\omega \sim 90^{\circ}$ or 270° and $60^{\circ} \lesssim i \lesssim 120^{\circ}$. We explore the resonance structure in the space (a, i) by means of dynamical maps and we find structures similar to those of space (a, e).

1. Introduction

Orbital resonances are an essential mechanism in the dynamics of minor bodies, planetary rings, satellite systems and planetary systems and they represent a fundamental core of knowledge of celestial mechanics. In this paper we will focus on the case of a small body in mean motion resonance (hereafter MMR) with a planet, with the aim of extending our understanding of its dynamics towards regions of the space of orbital elements that have not yet been fully explored. We recall that a particle with mean motion n is in the MMR k_p:k with a planet with mean motion n_p when the approximate relation $k_p n_p - kn \sim 0$ is satisfied, being k_p and k positive integers. The resonance is not limited to an exact value of semimajor axis a, on the contrary the resonance has some width in astronomical units (au) centered on the *nominal* position, a_0, deduced from $n = n_p k_p / k$. The picture astronomers have outlined along the years about resonant behavior is based, with few exceptions, on theories developed for low inclination orbits. These theories showed that the resonance domain in semimajor axis grows with the orbital eccentricity e: it goes from zero for $e = 0$ to wide regions for high e. In the case of the resonances with the giant planets of the Solar System, the resonant islands at high e are so wide that a large chaotic region is formed, due to the superposition of the different resonances. There is a very complete literature about MMRs, we can mention for example some chapters of books (Murray and Dermott, 1999; Morbidelli, 2002; Ferraz-Mello, 2007; Lemaître, 2010) and some reviews (Peale, 1976; Malhotra, 1998; Nesvorný et al., 2002; Gallardo, 2018).

From basic theories, we know that the orbital dynamics of a small body in resonance with a planet is defined by the disturbing function R (a, σ), where σ is the *critical angle* that we will define later. The equations of motion can be derived from its Hamiltonian $\mathcal{K}$, that can be found in the Appendix. The disturbing function R actually depends also on the other orbital parameters of the small body, but their typical evolution timescale is generally much larger than a and σ. All along this paper, we will focus only on the resonant (or semi-secular) timescale, over which (e, i, ω, Ω) can be considered fixed. The resonant motion imposes oscillations (called *librations*) of σ around an equilibrium value σ_0, correlated to oscillations of the semimajor axis a, though its value remains between limits defined by the borders (or *separatrices*) of the resonance (Nesvorný et al., 2002). The interval between these limits is called *width* of the resonance. Simplified analytical theories based on a unique resonant perturbing term of the form $R = A\cos(\sigma)$ usually call *strength* the coefficient A. The simplified Hamiltonian adopts a pendulum-like form and then the strength A is thus equal to the depth of the resonance island, whereas its width is proportional to $\sqrt{A}$. The overall geometry of the resonance is given by the level curves of $\mathcal{K}$ in the plane (a, σ). Of course, the remaining orbital elements (e, i, ω, Ω) are actually not exactly fixed. For example, we show in Fig. 1 the time evolution of a, e, i and σ of a test particle evolving inside the 3:1 resonance with Jupiter. The pendulum-like oscillations of a and σ are obvious. Their repercussions on e and i are insignificant compared to their long-term drift (not shown and not studied in this paper): we note in particular that the oscillations of e and i are exactly in phase with a, reflecting the fact that they are only a by-product of the coordinates used and not independent features of the dynamics.

The theories developed for low inclination orbits showed that in the low-eccentricity regime the strength of the resonance k_p:k is

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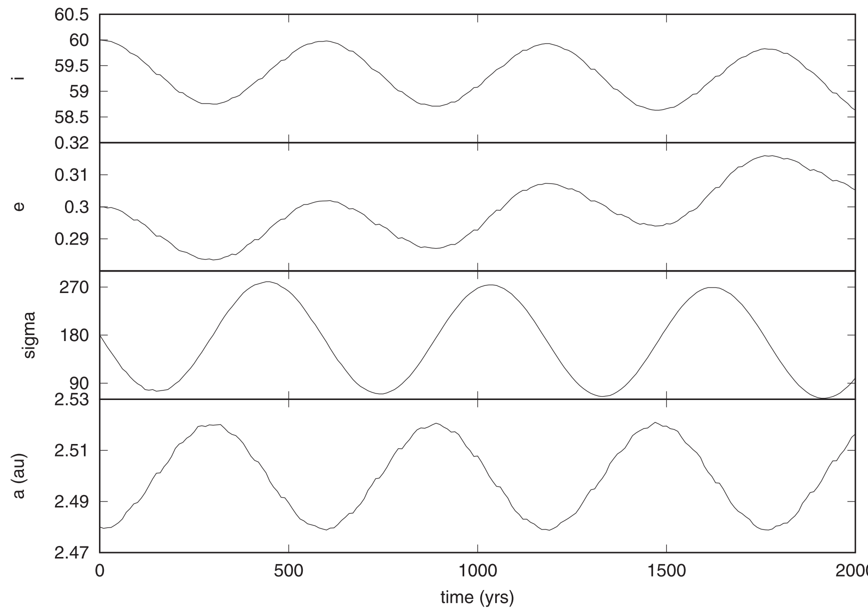

Fig. 1. Time evolution of a test particle inside the resonance 3:1 with Jupiter. The full set of orbital elements (*a, e, i*) oscillates (or librates) due to the resonance. The critical angle is $\sigma = 3\lambda_J - \lambda - 2\varpi$. A drift much slower than the resonant oscillations is also clearly visible, as a result of the secular dynamics inside the resonance. These long-term variations can have a much larger amplitude for *e* and *i* than the small resonant oscillations, possibly leading the particle towards chaotic regions.

proportional to e^q being *e* the eccentricity of the particle and $q = |k_p - k|$. So, *q* was conveniently called the *order* of the resonance. This justifies that only low order resonances have deserved the attention of astronomers. A complication to this simple rule was discovered by Morais and Giuppone (2012) and Morais and Namouni (2013). They demonstrated that for the extreme case of coplanar retrograde orbits (that means $i = 180°$) the strength of these resonances is not proportional to the eccentricity elevated to the power *q* but elevated to the power $|k_p + k|$. Being these integers both positive the order for retrograde orbits results to be always larger. Then, the difference between the integers factorizing both *n* is no longer representative of the order of the resonance for the full interval of orbital inclinations. Recently an analytical expansion for near polar orbits was obtained (Namouni and Morais, 2017) and it was found again a very different behavior: the expansion order of the disturbing function is not given by the value of *q* but by its parity: odd (1) or even (2). That expansion was recently extended to arbitrary inclinations by Namouni and Morais (2018). Their paper lists the terms up to fourth order terms in *e* and $\sin(i - i_r)$ where i_r is an arbitrary reference inclination.

In the general case, the leading-order terms of the disturbing function (including the so-called "pure eccentricity terms" of the classic expansions) are never proportional to *e* alone, but to coefficients of the type $e^N \sin i^M$, being *N* and *M* integers (Roig et al., 1998; Ellis and Murray, 2000; Namouni and Morais, 2018). This generates complicated expressions. Any analytical representation of the disturbing function is accurate only in a restricted domain of the orbital parameters, and the number of terms with non-negligible strength increases dramatically as we get further from the reference value around which the disturbing function is expanded. Then $R(\sigma)$ cannot be more represented by an unique term but the concept of strength can be generalized to the amplitude of the exact $R(\sigma)$ which in this case must be calculated numerically (Gallardo, 2006). Nevertheless, the concept of strength can still apply to a specific coefficient corresponding to some relevant critical angle as is done for example in Namouni and Morais (2018).

In numerical simulations of comets, centaurs and fictitious particles some works showed that captures in retrograde resonances are a common orbital state triggering the interest of the study of high inclination and retrograde resonances (Namouni and Morais, 2015; Fernández et al., 2016; Fernandez et al., 2018). In this context this paper generalizes the concept of strength to the full range of orbital elements and facilitates its calculation by a numerical procedure. We organize this paper as follows: in Section 2 we introduce the fundamental properties of the resonant motion, the numerical technique for computing the resonance strength, *SR*, for arbitrary resonances and we check *SR* with the existing theories and with purely numerical methods, mainly dynamical maps. In Section 3 we present a survey of the strengths in the space (*e, i, ω*) for some typical resonances still comparing the results to dynamical maps and we show some particular cases. In Section 4 we present the structure of MMRs in the space (*a, i*). We summarize the conclusions in Section 5.

2. Resonance strength

2.1. Notation

Different conventions have been utilized in the literature to describe the very simple relationship between the mean motions of two resonant objects. In this paper, we will call resonance $k_p{:}k$ the resonance generated by the commensurability given by $k_p n_p - kn \sim 0$. For example, 3:1 is a resonance interior to the perturbing planet and 1:3 is an exterior resonance. Following for example Ellis and Murray (2000), the resonant disturbing function, $R(\sigma)$, can be written as a series expansion of cosines which arguments are of the type

$$\sigma = k_p \lambda_p - k\lambda + \gamma \tag{1}$$

where λ_p and λ are the quick varying mean longitudes of the planet and particle respectively and γ is a slow evolving angle defined by a linear combination of the longitudes of the ascending nodes and longitudes of the perihelia of the particle and the planet involved. In the simplified case of a perturbing planet with zero inclination and circular orbit γ only depends on the asteroid's longitude of perihelion ϖ and longitude of the ascending node Ω (Gallardo, 2006; Morais and Namouni, 2013). Different linear combinations of ϖ and Ω generate different γ and consequently different σ, but all of them include the angle $k_p \lambda_p - k\lambda$, characteristic of the resonance $k_p{:}k$. All possible σ can be called critical angle but in general there is one particular σ that correlates better with