NRIAG Journal of Astronomy and Geophysics 7 (2018) 10–14

Contents lists available at ScienceDirect

NRIAG Journal of Astronomy and Geophysics

journal homepage: www.elsevier.com/locate/nrjag

Full length article

Studying the variation of eddy diffusivity on the behavior of advection-diffusion equation

Khaled Sadek Mohamed Essa[a,*], Aziz Nazer Mina[b], Hany Saleh Hamdy[c], Fawzia Mubarak[d], Ayman Ali khalifa[e]

[a] Environmental Physics, Department of Mathematics and Theoretical Physics, Nuclear Research Center, Egyptian Atomic Energy Authority, Cairo, Egypt
[b] Nuclear Physics, Department of Physics, Faculty of Science, Beni-suef University, Egypt
[c] Solid State Physics, Department of Physics, Faculty of Science, Beni-suef University, Egypt
[d] Radiation Protection Department, Nuclear Research Center, Egyptian Atomic Energy Authority, Cairo, Egypt
[e] Department of Physics, Faculty of Science, Beni-suef University, Egypt

ARTICLE INFO

Keywords:
Diffusion Equation
Separation of variables
Gaussian model
Laplace technique
Eddy diffusivity

ABSTRACT

In this work, the advection-diffusion equation was solved in two dimensions to calculate the normalized crosswind integrated concentration by Laplace technique. Considering that the wind speed is constant and we have two models of the vertical eddy diffusivity, one depends on downwind distance and the other model depends on vertical distance. A comparison between our proposed two models, Gaussian, previous work and observed data measured at Copenhagen, Denmark, have been carried out. One finds that there is a good agreement between predicted (2) model and the observed concentrations than predicted (1), Gaussian and previous work.

From the statistical technique, one finds that all models are inside a factor of two with observed data. Regarding to Normalized mean square error (NMSE) and Fraction Bais (FB), proposed model (2) is performance well with observed data than the predicted (1), Gaussian and previous work in unstable condition.

1. Introduction

It is very important to be aware of how contaminants are dispersed through the atmosphere. Unfortunately, Air pollutants influence directly or indirectly on man and environment. Essa and El-Otaify (2008), Alharbi (2011) discussed the dispersion of pollutant mainly depends on meteorological and topographical conditions. In order to understand the dispersion of contaminants in the atmosphere we should study physics that describes the transport of these contaminants in the atmosphere in different boundary conditions. Logan (2001), Mazaher et al. (2013), Scott and Gerhard (2005), Essa et al. (2014) and Tirabassi et al. (2010) studied advection-diffusion equation which depends on Gaussian and non-Gaussian solutions.

Amruta and Pradhan (2013) solved advection-diffusion equation under various circumstances and using various methods.

In this work we solved the advection-diffusion equation in two dimensions to obtain normalized integrated crosswind concentration using Laplace technique. Two models of the vertical eddy diffusivity were developed, considering constant wind speed. One of them depends on downwind distance and the other depends on vertical distance. Comparisons between them, Gaussian, previous work (Sharan and Modani, 2006) and observed data measured at Copenhagen, Denmark were carried out (Gryning and Lyck, 1984, Gryning et al. 1987).

2. Mathematical models

Diffusion equation is the most important in studying of pollutants dispersion into the atmosphere by using the gradient transport theory, this diffusion equation of pollutants in air can be written as (Tiziano and Vilhena, 2012, Tirabassi et al., 2008, 2009):

$$u\frac{\partial c(x,y,z)}{\partial x} = \frac{\partial}{\partial y}\left(k_y\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z\frac{\partial c}{\partial z}\right) \quad (1)$$

where u is the wind speed (m/s), c(x,y,z) is the concentration of pollutant (g/m^3), K_y, k_z are the eddy diffusivities in lateral and vertical direction respectively.

Peer review under responsibility of National Research Institute of Astronomy and Geophysics.

* Corresponding author.
E-mail address: mohamedksm56@yahoo.com (K.S.M. Essa).

https://doi.org/10.1016/j.nrjag.2018.02.003
Received 14 December 2017; Received in revised form 31 January 2018; Accepted 7 February 2018
Available online 22 February 2018

2.1. First model

In this model, one supposes that the vertical eddy diffusivity is a function of downwind distance i.e. $k_z = k(x)$. Integrating equation (1) with respect to y from $-\infty$ to ∞, then:

$$u\frac{\partial}{\partial x}\int_{-\infty}^{\infty} c(x,y,z)dy = k_y\frac{\partial c(x,y,z)}{\partial y}\Big|_{-\infty}^{\infty} + k(x)\frac{\partial^2}{\partial z^2}\int_{-\infty}^{\infty} c(x,y,z)dy \tag{2}$$

By supposing that:

$$\int_{-\infty}^{\infty} c(x,y,z)dy = c_y(x,z) \tag{3}$$

One gets that:

$$k_y\frac{\partial c(x,y,z)}{\partial y}\Big|_{-\infty}^{\infty} = 0 \tag{4}$$

By substituting from Eqs, (3) and (4) into Eq. (2), one gets:

$$u\frac{\partial c_y(x,z)}{\partial x} = \frac{\partial}{\partial z}\left[k(x)\frac{\partial c_y(x,z)}{\partial z}\right] \tag{5}$$

In first case the eddy diffusivity is supposed to be constant as a function of x.

$$u\frac{\partial c_{y(x,z)}}{\partial x} = k(x)\frac{\partial^2 c_y(x,z)}{\partial z^2} \tag{6}$$

Eq. (6) is solved under the boundary conditions:

(i) $c_y(0,z) = \frac{Q}{u}\delta(z-h_s)$, where h_s is a stack height
(ii) $c_y(x,z) = 0$ at $x,z \to \infty$
(iii) $k_z\frac{\partial c_y}{\partial z} = 0$ at $z = 0, z_i$

where z_i is the mixing height.

Taking $k(x) = \alpha\bar{u}x$, where α is the turbulence parameter such that: $\alpha = \left(\frac{\sigma_w}{u}\right)^2$, σ_w is the vertical velocity standard deviation (Moreira et al., 2014; Essa et al., 2007; Torbern, 2012, Pramod and Sharan, 2016).

$$\therefore k(x) = \frac{\sigma_w^2}{u}x$$

Taking Laplace transform on x as follows:

$$\tilde{c}(s,z) = \int_0^{\infty} c_y(x,z)e^{-sx}dx \tag{7}$$

Eq. (6) becomes:

$$\int_0^{\infty} u\frac{\partial c_y}{\partial x}e^{-sx}dx = \int_0^{\infty}\frac{\sigma_w^2 x}{u}\frac{\partial^2 c_y}{\partial z^2}e^{-sx}dx \tag{8}$$

Integrating Eq. (8), one gets:

$$-uc_y(0,z) + su\tilde{c}_y(s,z) = \frac{\sigma_w^2}{su}\frac{\partial^2\tilde{c}_y(s,z)}{\partial z^2} \tag{9}$$

Condition (i) is applied in Eq. (9) then.

$$\frac{\sigma_w^2}{su}\frac{\partial^2\tilde{c}_y(s,z)}{\partial z^2} - su\tilde{c}_y(s,z) = -Q\delta(z-h_s) \tag{10}$$

Now applying Laplace transform on z one gets:

$$\int_0^{\infty} e^{-pz}\frac{\sigma_w^2}{su}\frac{\partial^2\tilde{c}_y(s,z)}{\partial z^2}dz - \int_0^{\infty} sue^{-pz}\tilde{c}_y(s,z)dz = -\int_0^{\infty} e^{-pz}Q\delta(z-h_s)dz \tag{11}$$

$$\frac{\sigma_w^2}{su}\left[p^2\tilde{\tilde{c}}_y(s,p) - pc_y(s,0) - \frac{\partial\tilde{c}_y(s,0)}{\partial z}\right] - us\tilde{\tilde{c}}_y(s,p) = -Q\int_0^{\infty} e^{-pz}\delta(z-h_s)dz \tag{12}$$

After application the condition (iii), Eq. (12) becomes:

$$\frac{\sigma_w^2}{su}\left[p^2\tilde{\tilde{c}}_y(s,p) - pc_y(s,0)\right] - us\tilde{\tilde{c}}_y(s,p) = -Qe^{-ph_s} \tag{13}$$

$$\tilde{\tilde{c}}_y(s,p) = \frac{\frac{\sigma_w^2}{su}c_y(s,0)p}{\left(\frac{\sigma_w^2}{su}p^2 - us\right)} - \frac{Qe^{-ph_s}}{\left(\frac{\sigma_w^2}{su}p^2 - us\right)} \tag{14}$$

$$\tilde{\tilde{c}}_y(s,p) = \frac{c_y(s,0)p}{(p^2 - \frac{u^2s^2}{\sigma_w^2})} - \frac{Qe^{-ph_s}}{(p^2 - \frac{u^2s^2}{\sigma_w^2})} \tag{14}$$

$$\tilde{\tilde{c}}_y(s,p) = c_y(s,0)F(s,p) - Qe^{-ph_s}G(s,p) \tag{15}$$

where $F(s,p) = \frac{p}{(p^2 - \frac{u^2s^2}{\sigma_w^2})}$ and $G(s,p) = \frac{su/\sigma_w^2}{(p^2 - \frac{u^2s^2}{\sigma_w^2})}$

Take the inverse of Laplace transform on "z" i.e. $\mathscr{L}^{-1}\{(\tilde{\tilde{c}}_y(s,p),z\} = \tilde{c}_y(s,z)$

$$\tilde{c}_y(s,z) = \frac{c_y(s,0)}{2}\left[e^{\frac{su}{\sigma_w}z} + e^{-\frac{su}{\sigma_w}z}\right] - \frac{Q}{2}\frac{\sigma_w}{su}\left[e^{\frac{su}{\sigma_w}(z-h_s)} - e^{-\frac{su}{\sigma_w}(z-h_s)}\right]H(z-h_s) \tag{16}$$

where H is a Heaviside function.

Let $R_n = \frac{su}{\sigma_w}$

$$\tilde{c}_y(s,z) = \frac{c_y(s,0)}{2}[e^{R_nz} + e^{-R_nz}] - \frac{Q}{2R_n}[e^{R_n(z-h_s)} - e^{-R_n(z-h_s)}]H(z-h_s) \tag{17}$$

$$\tilde{c}_y(s,z) = c_y(s,0)\cosh R_nz - \frac{Q}{R_n}\sinh R_n(z-h_s) * H(z-h_s) \tag{18}$$

Using the boundary condition (iii) one gets:
$k_z\frac{\partial}{\partial z}\tilde{c}_y(s,z) = 0$ at $z = z_i$ then:

$$\frac{\partial}{\partial z}\tilde{c}_y(s,z) = R_nc_y(s,0)\sinh R_nz - \frac{Q}{R_n}R_n\cosh R_n(z-h_s)H(z-h_s) - \frac{Q}{R_n}\sinh R_n(z-h_s)\frac{\partial}{\partial z}H(z-h_s) \tag{19}$$

$$c_y(s,0)\sinh(R_nz_i) = \frac{Q}{R_n}\cosh(R_n(z_i-h_s))H(z_i-h_s) \tag{20}$$

$$c_y(s,0) = \frac{Q}{R_n}\frac{\cosh R_n(z_i-h_s)}{\sinh(R_nz_i)}$$

$$\therefore c_y(s,0) = \frac{Q}{\frac{\sigma_w}{su}}\frac{\cosh\frac{\sigma_w}{su}(z_i-h_s)}{\sinh\frac{\sigma_w}{su}z_i} \tag{21}$$

Substituting from Eq. (21) in Eq. (18) one gets:

$$\tilde{c}_y(s,z) = \frac{Q}{\frac{\sigma_w}{su}}\frac{\cosh\frac{\sigma_w}{su}(z_i-h_s)}{\sinh\frac{\sigma_w}{su}z_i}\cosh\left(\frac{\sigma_w}{su}z\right) - \frac{Q}{\frac{\sigma_w}{su}}\sinh\frac{\sigma_w}{su}(z-h_s) * H(z-h_s) \tag{22}$$

At ground level (i.e. z = 0), H (z-h_s) = 0, the crosswind integrated concentration can be written as follows:

$$\tilde{c}_y(s,0) = \frac{Q}{\frac{\sigma_w}{su}}\frac{\cosh\frac{\sigma_w}{su}(z_i-h_s)}{\sinh\frac{\sigma_w}{su}z_i} \quad \text{at } z = 0$$

By using Gaussian quadrature formulas then:

$$\frac{c_y(x,0)}{Q} = \sum_{i=1}^{N=8} a_i\left(\frac{s_i}{x}\right)\frac{1}{\frac{us_i}{x\sigma_w}}\frac{\cosh\frac{us_i}{x\sigma_w}(z_i-h_s)}{\sinh\frac{us_i}{x\sigma_w}z_i} \tag{23}$$

where N is the number of quadrature points. a_i and s_i is the Gaussian quadrature parameters.

2.2. Second Model

In the second model the eddy diffusivity is influenced by the vertical height (z), and then Eq. (6) can be written as: