Alexandria Engineering Journal (2016) 55, 1647–1653

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Alexandria Engineering Journal

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ORIGINAL ARTICLE

Subdivision depth for triangular surfaces ☆

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Received 1 December 2014; revised 26 January 2016; accepted 20 February 2016
Available online 10 March 2016

KEYWORDS

Subdivision surfaces;
Control polygon;
B-spline;
Triangular surface;
Error bound;
Subdivision depth

Abstract The aim of this attempt was to present an efficient algorithm for the evaluation of error bound of triangular subdivision surfaces. The error estimation technique is based on first order difference and this process is independent of parametrization. This technique can be easily generalized to higher arity triangular surfaces. The estimated error bound is expressed in-terms of initial control point sequence and constants. Here, we efficiently estimate error bound between triangular surface and its control polygon after *k*-fold subdivision and further extended to evaluate subdivision depth of the scheme.

1. Introduction

Subdivision is a simple and popular method to generate smooth limit curves and surfaces from discrete set of data points. It is an iterative algorithm, which is based on simple refinement rules to generate increasingly dense sequence of points under suitable hypothesis, converging to a continuous and smooth function. Starting from an initial control polygon, a subdivision scheme refers the computed values at the previous step according to the subdivision rules. The scheme is said to be convergent if there exists a limit curve. Efficiency of subdivision schemes is their flexibility and simplicity and they found their way into wide range of applications in computer graphics, medical imaging, industrial design and automotive design, etc. [1–3].

Triangular surfaces [4] are one of the fundamental paradigms of Computer Aided Geometric Design (CAGD). These are defined by de Boor nets and have a regular triangular structure. This class of triangular surfaces shares the properties of univariate [5] and tensor product B-splines [6]. The procedure for subdividing triangular surfaces exactly parallels the subdivision for tensor product B-spline surfaces. Actually, these are extension of B-splines surfaces.

For many applications such as rendering, intersection testing or design, it is important to know, how well the control polygon approximate the exact curve or a surface. In the last decade several researchers attempt to answer the question and to improve the rule to estimate error bounds. The techniques presented in [7–11] for computation of error bounds are based on parametrization, so they cannot be generalized to subdivision surfaces easily, methods presented in [12–14] are based on eigen analysis. Zeng and Chen [15] introduced the concept of neighbor points and by using the first-order difference of control points of Catmull–Clark surfaces, they

☆ This work is under the project of Higher Education Commission (HEC), Pakistan.

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Peer review under responsibility of Faculty of Engineering, Alexandria University.

http://dx.doi.org/10.1016/j.aej.2016.02.013
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obtained the rate of convergence of control meshes of Catmull–Clark surface. From the result of convergence, they derived a computational formula of subdivision depth for Catmull–Clark surfaces. Cheng and Yong [16] introduced computational formula for subdivision depth, which is based on second order forward differences for extra-ordinary Catmull–Clark subdivision surface patches. Mustafa et al. [17–23] have estimated error bound for binary, ternary, quaternary, non-stationary, n-ary curve, surface and volumetric model in-terms of maximal first order differences of the initial control point sequence and constants that depend on the subdivision mask. Huang et al. [24] derive a bound on the distance between a Catmull Clark subdivision surface patch and its limit face in terms of the maximum norm of the second order differences of the control points and a constant that depends only on the valence of the patch. Later on Mustafa et al. estimate the subdivision depth of Bajaj and $\sqrt{3}$ subdivision schemes for both regular and irregular patches [25,26]. Moncayo and Amat [27] presented error bounds for a class of subdivision schemes based on the two-scale refinement equation. In recent years Hashmi et al. [28] estimated the subdivision depth for Li subdivision scheme for regular and irregular patches.

In the present literature survey, it is evident that no such attempt has been made to evaluate subdivision depth for triangular subdivision surfaces. In this paper author successfully articulates the formula for subdivision depth for triangular surfaces based on first order differences by using estimation techniques.

The rest of the paper is arranged in following fashion: Some definition and preliminary notations are given in Section 2. Section 3 is devoted for the proof of main result based on some preliminary results. Future research directions are given in Section 4. To maintain the presentation of paper as simple as possible for readers, notations and typical mathematical proof of basic results are provided in the Appendices.

2. Definition and notations

Let $p^k_{i,j} \in \mathbb{R}^N$, $i,j \in \mathbb{Z}$, denote a sequence of points in $\mathbb{R}^N$, $N \geqslant 2$, where k is a non-negative integer then binary subdivision process for triangular surfaces [1, pp. 14–19] in our context can be restated as

$$p^{k+1}_{i+(m+\alpha-1)/2,j+(m+\beta-1)/2} = \sum_{r=-m+1}^{m-1}\sum_{s=-m+1}^{m-1}\sum_{l=0}^{m} a_{\alpha,r,l}a_{\beta,s,l}d_{m,l}p^k_{i+r,j+s}, \tag{2.1}$$

where $\alpha,\beta \in \{0,1\}$ or $\{1,2\}$, m is greater than 2, $a_{\alpha,j,l}$ and $d_{m,l}$ are defined by

$$a_{\alpha,j,l} = 2^{-m}\binom{m}{2(\frac{m+\alpha-1}{2}-j)-l}, \quad d_{m,l} = 2^{-m+2}\binom{m}{l},$$

for $\alpha = 0,1,2$, $j = -m+1,\ldots,m-1$, $l = 0,\ldots,m$, called subdivision mask. It is cautioned that (2.1) depends on labeling of the control polygon. For example for $m = 2$, 3, and 4, labeling of old and new points (A,B,C,D,E,F,G) is shown in Figs. 1 (a) and (b) and 2 respectively.

Given initial values $p^0_{i,j} \in \mathbb{R}^N$, $i,j \in \mathbb{Z}$, then in the limit $k \to \infty$, the process defines an infinite set of points in $\mathbb{R}^{\mathbb{N}}$. A necessary condition for the convergence of the subdivision process (2.1) for arbitrary initial data is that

$$\sum_{r=-m+1}^{m-1}\sum_{s=-m+1}^{m-1}\sum_{l=0}^{m} a_{\alpha,r,l}a_{\beta,s,l}d_{m,l} = 1, \tag{2.2}$$

where $\alpha,\beta \in \{0,1\}$ or $\{1,2\}$.

Let us suppose

$$\beta^k_t = \max_{i,j}\left\|\Delta^k_{i,j,t}\right\|, \quad k \geqslant 0, \quad t = 1,2, \tag{2.3}$$

where

$$\begin{cases} \Delta^k_{i,j,1} = p^k_{i+r+1,j+s} - p^k_{i+r,j+s}, & \forall\ r,s \in \mathbb{Z}, \\ \Delta^k_{i,j,2} = p^k_{i+s,j+r+1} - p^k_{i+s,j+r}, & \forall\ r,s \in \mathbb{Z}. \end{cases} \tag{2.4}$$

Suppose for $\alpha = 0,1,2$,

$$\begin{cases} \xi^1_{\alpha,l} = \sum_{p=1}^{m-1} a_{\alpha,p,l}, \quad \xi^2_{\alpha,l} = \sum_{p=-1}^{-m+1} a_{\alpha,p,l}, \quad \xi^3_{\alpha,l} = \sum_{r=-m+1}^{m-1} a_{\alpha,r,l}, \\ \xi^4_{\alpha,l} = \sum_{q=1}^{m-2}\tilde{a}_{\alpha,q,l} - \sum_{q=-2}^{-m+1}\tilde{\tilde{a}}_{\alpha,q+1,l}, \end{cases} \tag{2.5}$$

where

$$\tilde{a}_{\alpha,q,l} = \sum_{j=q+1}^{m} a_{\alpha,j,l}, \quad \tilde{\tilde{a}}_{\alpha,q+1,l} = \sum_{j=q}^{-m+1} a_{\alpha,j,l}.$$

Suppose further that

$$M^k_{(\alpha,\beta)} = \max_{i,j}\left\|p^{k+1}_{i+(m+\alpha-1)/2,j+(m+\beta-1)/2} - \frac{1}{2}\left(p^k_{i,j} + p^k_{i+\alpha-1,j+\beta-1}\right)\right\|, \tag{2.6}$$

where $\alpha,\beta \in \{0,1\}$ or $\{1,2\}$.

Also

$$\delta = \max\left\{\begin{array}{l} \left|\sum_{l=0}^{m}\sum_{s=-m+1}^{m-1}\sum_{r=-m+1}^{m-2} a_{1,s,l}d_{m,l}e_{r,l}\right|, \left|\sum_{l=0}^{m}\sum_{s=-m+1}^{m-1}\sum_{r=-m+1}^{m-2} a_{1,s,l}d_{m,l}f_{r,l}\right|, \\ \left|\sum_{l=0}^{m}\sum_{s=-m+1}^{m-1}\sum_{r=-m+1}^{m-2} a_{2,s,l}d_{m,l}e_{r,l}\right|, \left|\sum_{l=0}^{m}\sum_{s=-m+1}^{m-1}\sum_{r=-m+1}^{m-2} a_{0,s,l}d_{m,l}f_{r,l}\right|, \end{array}\right\} \tag{2.7}$$

where $e_{r,l} = \sum_{p=-m+1}^{r}(a_{1,p,l} - a_{2,p,l})$ and $f_{r,l} = \sum_{p=-m+1}^{r}(a_{0,p,l} - a_{1,p,l})$.

Rest of the notations are in Appendix A.

2.1. Subdivision depth

Given control polygon of n-ary subdivision surface and an error tolerance ϵ, if we subdivide control polygon k times, so that the error between resulting polygon and subdivision surface is smaller than ϵ, then k is called subdivision depth of subdivision surface with respect to ϵ.

3. The error bounds for triangular surfaces

In this Section, the main result for error bounds is presented for triangular surfaces, which is based on some preliminary results. Finally, the section ends on subdivision depth formula.

Lemma 3.1. *Given initial triangular control polygon $p^0_{i,j} = p_{i,j}$, $i,j \in \mathbb{Z}$, let the values $p^k_{i,j}, k \geqslant 0$ be defined recursively by subdivision process (2.1) together with (2.2) then*