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ORIGINAL ARTICLE

Numerical simulation of a fractional model of temperature distribution and heat flux in the semi infinite solid

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Abstract In this paper, a fractional model for the computation of temperature and heat flux distribution in a semi-infinite solid is discussed which is subjected to spatially decomposing, time-dependent laser source. The apt dimensionless parameters are identified and the reduced temperature and heat flux as a function of these parameters are presented in a numerical form. Some special cases of practical interest are also discussed. The solution is derived by the application of the Laplace transform, the Fourier sine transform and their derivatives. Also, we developed an alternative solution of it by using the Sumudu transform, the Fourier transform and their derivatives. These results are received in compact and graceful forms in terms of the generalized Mittag-Leffler function, which are suitable for numerical computation.

1. Introduction

In the modeling of many physical and chemical processes and engineering systems fractional differentiation has been widely used. The instances are electrochemistry and electromagnetic waves, diffusion waves, fractal electrical networks, electrical machines, nanotechnology, viscoelastic supplies and systems, quantum evolution of complex systems [1], and heat conduction [2]. Automatic control is also a field in which many applications of fractional differentiations have been anticipated. Recently, it is demonstrated that the real state of a fractional order system is not exactly observable [3]. However, the authors have also have demonstrated that the pseudo state vector of the pseudo state space description can be estimated using a Luenberger like observer. As fractional order derivatives and integrals explain the memory and genetic properties of different substances, the above mentioned new models are more sufficient than the earlier used integer order models [4]. This is the biggest advantage of the fractional order models in comparison with integer order models in which such effects are neglected. A semi-infinite solid is an idealized body that has a single plane surface and extends to infinity in all directions. This idealized body is used to specify that the temperature

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change in the part of the body in which we are interested is due to the thermal situation on a single surface. The earth, for instance, can be considered as a semi-infinite medium in determining the variation of temperature close to its surface. A thick wall can also be modeled as a semi-infinite medium if we are interested in the variation of temperature in the region near one of the surfaces, and the other surface is extreme to have any impact on the region of interest during the time of surveillance. In view of great importance of fractional differential equations many authors have paid attention for handling linear and nonlinear fractional differential equations [5–7]. In recent years many authors have employed various analytical schemes to investigate nonlinear problems arising in scientific and technological fields such as nonlinear oscillation of a centrifugal governor system [8], dynamic analysis of generalized conservative nonlinear oscillators [9], nonlinear vibrating systems [10], and frequency analysis of strongly nonlinear generalized duffing oscillators [11].

2. Preliminary results

Consider semi-infinite solid initially at temperature T_0. The left face of the solid is suddenly raised to temperature T_s at time zero and defining $\theta = \frac{T-T_0}{T_s-T_0}$. If we suppose, constant thermal conductivity, no internal heat generation and insignificant temperature variation in the y and z directions. The relevant differential equation is given by classical non-homogenous heat equation defined by [12]:

$$\frac{\partial \theta}{\partial t} = C\frac{\partial^2 \theta}{\partial x^2}, \tag{1}$$

where K is the thermal diffusivity. Subject to boundary conditions are

$$\left.\begin{array}{l} t=0;\ \theta=0 \\ x=0;\ \theta=1 \\ x\to\infty;\ \theta\to 0 \end{array}\right\} \tag{2}$$

The following well-known facts are considered to study the temperature distribution and heat flux in the semi infinite solid.

The Laplace transform is defined by [13]

$$L\{f(x)\} = \int_0^\infty e^{-st} f(t)dt; \quad Re(s) > 0. \tag{3}$$

The Fourier sine transform is defined by [14]

$$\overline{F}(n,\ t) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x,\ t)\sin nx\, dx. \tag{4}$$

The error function of x is defined by [15]

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-z^2)dz \tag{5}$$

and the complimentary error function of x is defined as

$$erf_c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-z^2)dz. \tag{6}$$

A generalization of the Mittag-Leffler function [16,17]

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+1)}, (\alpha \in C, \mathbb{R}(\alpha) > 0) \tag{7}$$

was introduced by [18] in the general form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+\beta)}, (\alpha, \beta \in C, \mathbb{R}(\alpha) > 0) \tag{8}$$

also derived by [19] in the following integral

$$\int_0^\infty e^{-st} t^{\beta-1} \frac{d^k}{dz^k} E_{\alpha,\beta}(xt^\alpha)dt = \frac{k!s^{\alpha-\beta}}{(s^\alpha - x)^{k+1}}. \tag{9}$$

The fractional derivative of order $\alpha > 0$ is presented by Caputo [20] in the form

$$\begin{aligned} {}_0^c D_x^\alpha f(x) &= \frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha-m+1}}d\tau,\ m-1<\alpha<m \\ &= \frac{d^m f(t)}{dt^m}, if\ \alpha = m; m \in N \end{aligned} \tag{10}$$

where $\frac{d^m f(t)}{dt^m}$ is the mth derivative of order m of the function $f(t)$ with respect to t. The Laplace transform of this derivative is given by [4]

$$L\left[{}_0^c D_x^\alpha f(x); s\right] = s^\alpha \bar{f}(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0+),\ m-1<\alpha\le m. \tag{11}$$

A generalization of the Caputo fractional derivative operator Eq. (10) is given by [21], by introducing a right-sided fractional derivative operator of two parameters of order $0 < \alpha < 1$ and $0 \leqslant \beta \leqslant 1$ as

$${}_0 D_{a+}^{\alpha,\beta} f(x) = I_{a+}^{\beta(1-\alpha)} \frac{d}{dx}\left(I_{a+}^{(1-\beta)(1-\alpha)} f(x)\right). \tag{12}$$

If we put $\beta = 1$, Eq. (12) reduces the Caputo fractional derivative operator assigned from Eq. (10).

Laplace transform formula for this operator [21] is given by

$$L\left[{}_0 D_x^{\alpha,\beta} f(x); s\right] = s^\alpha \tilde{f}(s) - s^{\beta(\alpha-1)} I_{0+}^{(1-\beta)(1-\alpha)} f(0+);\ 0<\alpha\le 1. \tag{13}$$

Sumudu transform formula for this operator [21,22], holds the formula

$$S\left[{}_0 D_x^{\alpha,\beta} f(x); s\right] = u^{-\alpha}\tilde{f}(u) - u^{-\beta(\alpha-1)+1} I_{0+}^{(1-\beta)(1-\alpha)} f(0+);\ 0<\alpha\le 1, \tag{14}$$

where the initial value term

$$I_{0+}^{(1-\beta)(1-\alpha)} f(0+), \tag{15}$$

involves the Riemann–Liouville fractional integral operator of order $(1-\beta)(1-\alpha)$ evaluated in the limit as $x \to 0+$. For more details and properties of this operator see in [23].

The simplest Wright function is defined by [24]

$$\mathcal{W}(\alpha, \beta; z) = \sum_{k=0}^\infty \frac{1}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}, \text{ where } \alpha, \beta, z \in C. \tag{16}$$

Generalized k-Wright function is an exciting generalization of Wright function Eq. (16). Some exciting properties of the generalized k-Wright function are obtained by [25].

Following integral [26] is required for simplification

$$\int_0^\infty n\ sinnx\ E_{\alpha,\alpha+1}\left(-n^2 Kt^\alpha\right)dn = \frac{\pi}{2Kt^\alpha}\mathcal{W}\left(-\frac{\alpha}{2}, 1; \frac{-x}{\sqrt{Kt^\alpha}}\right). \tag{17}$$