Physics Letters B 782 (2018) 94–98

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Topology of electroweak vacua

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ARTICLE INFO

Article history:
Received 12 March 2018
Received in revised form 20 April 2018
Accepted 6 May 2018
Available online 9 May 2018
Editor: B. Grinstein

ABSTRACT

In the Standard Model, the electroweak symmetry is broken by a complex, $SU(2)$-doublet Higgs field and the vacuum manifold $SU(2) \times U(1)/U(1)$ has the topology of a 3-sphere. We remark that there exist theoretical alternatives that are locally isomorphic, but in which the vacuum manifold is homeomorphic to an arbitrary non-trivial principal $U(1)$-bundle over a 2-sphere. These alternatives have non-trivial fundamental group and thus feature topologically-stable electroweak strings. An alternative based on the manifold $\mathbb{R}P^3$ (with fundamental group $\mathbb{Z}/2$) allows custodial protection of gauge boson masses and their couplings to fermions and has an explicit realisation in the Minimal Composite Higgs Model, in the case of maximal electroweak symmetry breaking. We show that, in common with all alternatives to S^3, such models have a problem with fermion masses.

1. Introduction

Decades of experiment have confirmed that the weak nuclear force and the electromagnetic force are described by a gauge theory in which a group locally isomorphic to $SU(2) \times U(1)$ is non-linearly realised in the vacuum, with only the electromagnetic subgroup $U(1) \subset SU(2) \times U(1)$ being linearly realised. Thus, the electroweak (EW) vacuum is degenerate and the vacua are described by a homogeneous space $SU(2) \times U(1)/U(1)$.

The starting point for this Article is the observation that there are many ways to include $U(1)$ in $SU(2) \times U(1)$; different ways lead to homogeneous spaces that can be topologically inequivalent. In the Standard Model (SM), the vacuum manifold arises due to a non-vanishing vacuum expection value (VEV) of the Higgs field, carrying the doublet representation of $SU(2)$, and is homeomorphic to the 3-sphere, S^3. As is well-known, this is rather boring from a physicist's point of view, since the vanishing of the homotopy groups $\pi_1(S^3)$ (respectively $\pi_2(S^3)$) implies the absence of topologically-stable strings (respectively monopoles). Here, we investigate all possible inclusions of $U(1)$, which lead to vacuum manifolds with fundamental group given by an arbitrary cyclic group; alternatives to the SM based on such inclusions thus feature topologically-stable strings, with potentially interesting consequences, *a priori*, for astrophysics, cosmology, and particle physics.

We begin our investigation with a study of the topological properties of the homogeneous spaces obtained via the different $U(1)$ inclusions. Informally, the topologically distinct inclusions of $U(1)$ differ in the number of times, p, that the $U(1)$ subgroup is wrapped around the $U(1)$ factor of $SU(2) \times U(1)$. We show that the homogeneous spaces are topologically equivalent to lens spaces and to circle bundles over a 2-sphere. Our first main result is to catalogue the basic algebraic topological invariants of the spaces, in particular showing that the fundamental group is isomorphic to the cyclic group with p elements, $\mathbb{Z}/p$. This implies that a physical model with such a vacuum structure (assuming that such a model exists) necessarily features stable topologically strings carrying a charge, labelled by an integer, that is conserved under addition modulo p. One expects that, in such a model, a network of such strings would form in the early Universe and persist to this day.

Having established that there exist an infinite number (one for each integer p) of novel possibilities for the vacuum manifold topology that are compatible with the usual local EW vacuum structure, we next turn to the question of whether there exist physical models featuring them. The answer is yes, with a rather obvious example being given by the non-linear sigma model with target space $SU(2) \times U(1)/U(1)$, which furnishes us with a consistent effective theory for the dynamics in the low-energy, broken-symmetry phase. We also show that one can construct linear sigma models with these vacuum manifolds, providing a description that is valid up to arbitrarily high energies; simple examples of such models are provided by theories in which the Higgs

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https://doi.org/10.1016/j.physletb.2018.05.013

field of the SM, which carries a doublet irreducible representation (irrep) of $SU(2)$, is replaced by a scalar particle carrying a higher-dimensional irrep.

Models of these types are rather implausible from the phenomenological viewpoint, but it turns out that one can also find non-trivial vacua in a model that is currently beloved by phenomenologists. This model is the so-called Minimal Composite Higgs Model, and is arguably the leading candidate for a model in which the EW symmetry is broken naturally via strong dynamics. At energies below the TeV scale, it may be described by a non-linear sigma model with target space isomorphic to $SO(5)/SO(4)$, at least locally. But, in order that the decay rate of Z bosons to bottom quarks not differ substantially from the experimental value, it is convenient to choose the target space to be globally $SO(5)/O(4)$, which is homeomorphic to $\mathbb{R}P^4$. As is well-known [1], the dynamics cannot be fully invariant under the $SO(5)$ action and the small breakings thereof (due, *e.g.*, to gauging of the electroweak subgroup and couplings to fermions) induce a potential for the scalar degrees of freedom on the target space. The EW vacuum manifold is then obtained as the minimum of this potential. With no EW symmetry breaking ('$v = 0$' in the usual notation of the composite Higgs literature), the vacuum manifold consists of a point, whereas with maximal EW symmetry breaking ('$v = f$' in the usual notation), it consists of the manifold $\mathbb{R}P^3$, with fundamental group $\mathbb{Z}/2$. In intermediate cases ('$0 < v < f$'), the vacuum manifold is the usual S^3 of the SM.

Although our current computational competence makes it impossible to make precise predictions in models featuring strong dynamics, we can nevertheless make estimates using dimensional analysis. As is well-known, in composite Higgs models, the measured S-parameter suggests that we need a vacuum in which $v/f \lesssim 0.2$, such that a vacuum with topological strings is ruled out. It is thus of interest to ask whether there are any models featuring EW topological strings that are phenomenologically viable, given current experimental data. We shall see that, whilst it is possible to reconcile such models with the panoply of precision measurements that have been performed in the EW sector (such as the mass ratios of gauge bosons and their couplings to fermions), they are all ultimately ruled out by the simple (although somewhat subtle in origin) fact that such models are incompatible with non-vanishing fermion masses.

Thus, we arrive at the conclusion that, whatever the true mechanism of EW symmetry breaking, it must lead to a vacuum manifold homeomorphic to S^3, just as in the SM, unless we are willing to entertain drastic modifications of the structure of the EW sector. Although this result is somewhat disappointing, it is perhaps hardly a surprise, given that there already exists so much data probing the local properties of EW vacuum manifold, all of which confirms the SM. Nevertheless, we feel that the very existence of such theoretical alternatives to the status quo, and their appearance in one of the most favoured models of natural EW symmetry breaking, is a noteworthy curiosity in its own right.

The outline is as follows. In §2, we discuss the topology of the vacuum manifold $SU(2) \times U(1)/U(1)$ and in §3 we show that an general effective field theory based on a vacuum manifold with non-trivial topology can be consistent with data, apart from a problem with fermion masses. In §4 and §5, we present explicit examples with non-trivial topology based on linear sigma models and composite Higgs models.

2. Topology of $SU(2) \times U(1)/U(1)$

We begin our discussion by assuming that the EW gauge group really is $G = SU(2) \times U(1)$, deferring discussion of groups locally isomorphic thereto until the end. We write elements of G as (U, z), where U is a 2×2 unitary matrix with unit determinant and z is a unit complex number. For $p, q \in \mathbb{Z}$ there is a homomorphism $\phi_{p,q} : U(1) \to G$ given by $\phi_{p,q}(z) = (\mathrm{diag}(z^q, z^{-q}), z^p)$, and if (p, q) are coprime then $\phi_{p,q}$ is injective, in which case we write $H_{p,q} \subseteq G$ for its image. (Any injective homomorphism $\phi : U(1) \to G$ is conjugate to some $\phi_{p,q}$, as its projection to the $SU(2)$-factor may be conjugated to land in the standard maximal torus.)

Our first goal is to investigate the topology of the homogeneous spaces $G/H_{p,q}$. An immediate result is that $G/H_{p,q}$ cannot be homeomorphic for different p, because a loop wound once around $H_{p,q}$ is wound p times around the $U(1)$ factor of G. This implies, using the long exact sequence of homotopy groups $\pi_1(H_{p,q}) \cong \mathbb{Z} \to \pi_1(G) \cong \mathbb{Z} \to \pi_1(G/H_{p,q}) \to \pi_0(H_{p,q}) \cong 0$ of the fibre bundle $H_{p,q} \hookrightarrow G \to G/H_{p,q}$, that $\pi_1(G/H_{p,q}) \cong \mathbb{Z}/p$. Moreover, we see that topologically-stable string configurations occur when $p \neq 1$ [2].

To investigate the topology further, let $K_{p,q} = H_{p,q} \cap (SU(2) \times \{1\}) \subseteq SU(2)$ and consider the function $\pi : A \mapsto (A, 1)H_{p,q} : SU(2) \to G/H_{p,q}$. This is a composition of smooth maps $SU(2) \hookrightarrow G \to G/H_{p,q}$ and so smooth. The differential at the identity $D\pi : \mathfrak{su}(2) \to \mathfrak{g}/\mathfrak{h}_{p,q}$ is an isomorphism, and by homogeneity it follows that π is a submersion, and hence a local diffeomorphism. Furthermore the right $K_{p,q}$-action on $SU(2)$ acts freely and transitively on the fibres of π, exhibiting it as a principal $K_{p,q}$-bundle, and hence giving a diffeomorphism $SU(2)/K_{p,q} \cong G/H_{p,q}$.

Now $K_{p,q} = \{\mathrm{diag}(e^{2\pi iqk/p}, e^{-2\pi iqk/p}) : k \in \mathbb{Z}\}$ is the same subgroup of $SU(2)$ as $K_{p,1}$, because (p, q) are coprime, and as $K_{-p,1}$: thus we shall suppose $p > 0$. It follows that $G/H_{p,q}$ is diffeomorphic to $SU(2)/K_{p,1}$, which is further diffeomorphic, as we now show, to a lens space [3]. These spaces are of great historical importance in mathematics, providing the first examples of manifolds whose homeomorphism type is determined by neither their fundamental group and homology [4], nor even their homotopy type [5]. The lens space $L(n, m)$ is defined for (n, m) coprime as the quotient of the unit sphere, $S^3 \subset \mathbb{C}^2$ by the free $\mathbb{Z}/n$-action generated by $(z_1, z_2) \mapsto (e^{2\pi i/n} z_1, e^{2\pi im/n} z_2)$. Identifying $SU(2)$ with the unit sphere $S^3 \subset \mathbb{C}^2$, $SU(2)/K_{p,1}$ is thus identified with the lens space $L(p, 1)$.

The lens spaces $L(p, 1)$ are precisely those 3-manifolds that arise as principal $U(1)$-bundles over the 2-sphere (except for $S^2 \times U(1)$). Indeed, the clutching construction shows that such bundles are in bijection with $\pi_1(U(1)) = \mathbb{Z}$, and this bijection may be given by assigning to a principal $U(1)$-bundle over the 2-sphere its Euler number. Writing $U(1) = \{\mathrm{diag}(e^{i\theta}, e^{-i\theta}) : \theta \in [0, 2\pi)\} \subseteq SU(2)$, the Hopf bundle $h_1 : SU(2) \to SU(2)/U(1) = S^2$ is the principal $U(1)$-bundle with Euler number 1. As $K_{p,1} \subseteq U(1)$ the map h_1 is the composition

$$SU(2) \longrightarrow SU(2)/K_{p,1} \xrightarrow{h_p} SU(2)/U(1) = S^2$$

of a p-fold covering map and a principal $(U(1)/K_{p,1} \cong U(1))$-bundle h_p, whose Euler number is therefore p and whose total space is $G/H_{p,q} \cong SU(2)/K_{p,1} \cong L(p, 1)$.

From this perspective, we may use standard results to read off the algebraic topological invariants of $G/H_{p,q}$: the homotopy groups are given by $\pi_1 = \mathbb{Z}/p, \pi_{i>1} = \pi_{i>1}(S^3)$ (so $\pi_2 = 0, \pi_3 = \mathbb{Z}, \pi_4 = \mathbb{Z}/2$, &c.); the integral cohomology is given by $H^0 = \mathbb{Z}, H^1 = 0, H^2 = \mathbb{Z}/p, H^3 = \mathbb{Z}$. Most interesting among these, for physicists, is $\pi_1 = \mathbb{Z}/p$.

How do these results relate to the SM? In that case, we postulate the existence of a Higgs field, that is a matter field ϕ whose potential is such that it acquires an non-vanishing VEV. It carries the doublet irreducible representation of $SU(2)$ and its charge $q \in \mathbb{Z}$ under $U(1)$ is non-vanishing, but otherwise arbitrary. The G-action is then $G : \phi \mapsto U z^q \phi$. Without loss of generality, we