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Screw-symmetric gravitational waves: A double copy of the vortex

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ABSTRACT

Plane gravitational waves can admit a sixth 'screw' isometry beyond the usual five. The same is true of plane electromagnetic waves. From the point of view of integrable systems, a sixth isometry would appear to over-constrain particle dynamics in such waves; we show here, though, that no effect of the sixth isometry is independent of those from the usual five. Many properties of particle dynamics in a screw-symmetric gravitational wave are also seen in a (non-plane-wave) electromagnetic vortex; we make this connection explicit, showing that the screw-symmetric gravitational wave is the classical double copy of the vortex.

1. Introduction

The plane wave approximation provides a simplified setting in which to investigate signatures of gravitational waves [1,2], such as the velocity memory effect [3–5], in which particles initially at rest acquire a constant, nonzero velocity after the wave has passed over them. The same effect is seen in electromagnetic plane waves, see e.g. [6], and [7] for historical references, with connections to the infra-red in both cases [8,7]. The possibility of mapping gravitational observables onto a simpler gauge theory setting [9,10] provides one motivation for studying the "classical double copy" [11], that is the mapping of classical solutions of Einstein's equations to classical solutions of Yang–Mills' equations. This is part of a larger program on colour-kinematic duality, or double copy, a precise conjecture about how scattering amplitudes in gravity can be obtained from those in gauge theory by replacing colour structure with kinematic structure [12–14]. The double copy conjecture has been proven at tree level, and there are an increasing number of nontrivial examples at loop level, see [15] for a review. In this context we note that plane waves provide a testing ground for extending the double copy programme to curved backgrounds [16].

There are noticeable similarities between particle motion in an electromagnetic vortex [17], which is not a plane wave, and in certain circularly polarised gravitational waves. The latter have been investigated as models of the waves emitted in various astrophysical phenomena [18,1,19]. Such waves can show an enlarged symmetry group containing an additional 'screw isometry' [20,4,5] beyond the five common to all plane waves. Our focus here is on the role played by this (and other) additional symmetries in charge motion, and our goal is to tie this to related results in integrable systems, to dynamics in electromagnetic vortices, and to the classical double copy.

This paper is organised as follows. In Sect. 2 we review the isometries of, and particle motion in, plane gravitational and electromagnetic waves. From the point of view of integrable systems these are rather special 'superintegrable' systems. In Sect. 3 we consider the screw isometry, which would seem to imply the existence of one conserved quantity too many. We resolve this, showing explicitly that the implied integral of motion is not independent of the other five. In Sect. 4 we compare charge motion in the screw-symmetric wave with that in an electromagnetic vortex [17], finding many similarities. We make the connection concrete by observing that the screw-symmetric wave is the classical double copy of the vortex. We discuss related cases and conclude in Sect. 5.

2. Isometries and (super)-integrable motion in plane waves

2.1. Gravitational plane waves

In order to make symmetries manifest we begin in Baldwin-Jeffery-Rosen (BJR) coordinates $\{u, v, x^j\}$, where the plane wave metric has the form [21,22]

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \mathrm{d}u\mathrm{d}v - \gamma_{ij}(u)\mathrm{d}x^i\mathrm{d}x^j\,, \qquad j\in\{1,2\}\,. \tag{1}$$

These coordinates are not global, and the γ_{ij} are constrained by the vacuum equations, but this will not affect our arguments. We will switch to globally defined coordinates later (see e.g. [7,16] for recent discussions and further references). The five Killing vectors of the metric are

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$$\left\{\frac{\partial}{\partial x^i}, 2x^i\frac{\partial}{\partial v}+G^{ij}(u)\frac{\partial}{\partial x^j}, \frac{\partial}{\partial v}\right\}, \quad \text{where} \quad G^{ij}(u)=\int^u \mathrm{d}s\,\gamma^{ij}(s), \tag{2}$$

corresponding to invariance under the Carroll group [23] with broken rotations [24]. Now consider a test particle in this background. Each Killing vector implies the existence of a conserved quantity in the particle motion. To analyse this we use the Hamiltonian formalism, which requires gauging the reparameterisation invariance of the particle action as usual, and we take u as time [25]. The action and Hamiltonian are

$$S=-m\int\sqrt{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu} \quad\longrightarrow\quad H(u)=\frac{\gamma^{ij}(u)p_ip_j+m^2}{4p_v}\,, \tag{3}$$

where $\{p_v, p_j\}$ are the respective conjugate momenta to $\{v, x^j\}$. The five conserved quantities corresponding to the five Killing vectors above are

$$\{Q_1,\cdots,Q_5\}:=\left\{p_j\,,\ 2x^ip_v+G^{ij}(u)p_j\,,\ p_v\right\}. \tag{4}$$

The conservation of these five is enough to determine all momenta and x^j algebraically, after which Hamilton's equation for v may be integrated directly. The question we want to address is, how many conserved quantities can there be? To answer this we need some general results on integrable systems.

An autonomous Hamiltonian system with $2n$-dimensional phase space is (polynomially) superintegrable if there exist $N>n$ independent phase space functions Q_j (polynomial in the momenta) which Poisson commute with the Hamiltonian (are conserved), and such that n of them are in involution, $\{Q_i, Q_j\}=0\,\forall\, i,j\in\{1\ldots n\}$. Systems with $N=2n-1$, the maximum possible number, are called maximally superintegrable [26,27]. While $2n-1$ conserved functions always exist *locally* [28], it is very rare to find systems in which they are globally defined polynomials in the momenta [27]. Superintegrable systems have many appealing properties; the classical equations of motion can admit an algebraic solution, and there is a conjecture that all corresponding quantum systems are exactly solvable [29].

A test particle in a gravitational plane wave is a superintegrable system; to show this, and noting that the Hamiltonian is time-dependent, we follow the standard method of converting to an autonomous system[1]; we expand phase space to eight dimensions by promoting u to a coordinate with conjugate momentum p_u, and use a new Hamiltonian $K=H-p_u$, for a review see [32]. Writing a dash for a derivative with respect to a new time (which appears nowhere explicitly), the time-derivative of any quantity Q is

$$Q'=\{K,Q\}_* \quad \text{where} \quad \{A,B\}_*=\frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial p_\mu}-\frac{\partial B}{\partial x^\mu}\frac{\partial A}{\partial p_\mu}\,. \tag{5}$$

In particular, we have as usual $u'=-\partial K/\partial p_u=1$. Now, in general there is no way to know *a priori* if a given system is (super)integrable. To derive the conserved quantities one can simply make an ansatz for Q (e.g. that it is quadratic in momenta) and impose (5); this yields a series of algebraic and differential equations determining the form of Q, see [27,33,34] for examples and references. In our plane wave case, this procedure yields $Q_1\ldots Q_5$ as in (4), along with two further conserved quantities; $Q_6=p_up_v-p_vH(u)$, which is just the mass-shell condition, and Q_7, given by

$$Q_7=4p_v^2v-m^2u-G^{ij}(u)p_ip_j\,. \tag{6}$$

These seven are functionally independent.[2] Thus we have the maximum number of seven independent conserved quantities, polynomial in the momenta. The system is maximally polynomially superintegrable. The solution of the equations of motion proceeds algebraically from here: the three momenta are conserved, Q_4 and Q_5 then determine $\{x^1, x^2\}$ as functions of time u, while Q_7 determines v.

2.2. Electromagnetic plane waves

Let us compare with electromagnetic plane waves. We work in lightfront coordinates $\{u, v, x^1, x^2\}$, the metric being (1) with $\gamma_{ij}(u)\to\delta_{ij}$. In order to make the connections with the gravitational case clear we represent an arbitrary electromagnetic plane wave $F_{\mu\nu}\equiv F_{\mu\nu}(u)$ using the two-component 'BJR' potential

$$A(x)=A_j(u)\mathrm{d}x^j\,, \qquad j\in\{1,2\}\,. \tag{8}$$

The particle action and, again taking u as time, (reparameterisation-)gauge-fixed Hamiltonian are now

$$S=-\int\mathrm{d}\tau\, m\sqrt{\dot{x}.\dot{x}}+\dot{x}.A(x) \quad\longrightarrow\quad H(u)=\frac{(p_j-A_j(u))^2+m^2}{4p_v}\,. \tag{9}$$

An arbitrary electromagnetic plane wave has five isometries, $\mathcal{L}_\xi F_{\mu\nu}=0$, for

$$\xi\in\left\{\frac{\partial}{\partial x^j}\,, 2x^j\frac{\partial}{\partial v}+u\frac{\partial}{\partial x^j}\,, \frac{\partial}{\partial v}\right\}, \tag{10}$$

corresponding to invariance under three translations and two null rotations respectively. These are of course in direct analogy to (2) and again span the Carroll group with broken rotations. Because these are Poincaré transformations they imply the existence of five conserved quantities; for $\xi\equiv\xi^\mu\partial_\mu$ Poincaré we have [34]

$$\begin{aligned}&\mathcal{L}_\xi F_{\mu\nu}=0 \implies Q\equiv\xi(x).p-\Lambda(x)=\text{constant},\\ &\quad\text{where } \mathcal{L}_\xi A_\mu=\partial_\mu\Lambda\,.\end{aligned} \tag{11}$$

(The functions Λ appear because the potential need only be symmetric up to U(1) gauge transformations.) The five conserved quantities following from the Poincaré symmetries of the plane wave (10) are

$$\begin{aligned}&\{Q_1,\cdots,Q_5\}=\left\{p_j\,,\ 2x^jp_v+up_j-G_j(u)\,,\ p_v\right\}\\ &\quad\text{for}\quad G_j(u)=\int^u\mathrm{d}s\,A_j(s)\,,\end{aligned} \tag{12}$$

in which the integrals are gauge terms Λ as in (11). These are again in analogy to (4). There are two further conserved quantities on expanded phase space; Q_6 is as above but with the Hamiltonian (3) replaced by (9), while

$$Q_7=4p_v^2v-(p_ip_i+m^2)u+2p_jG_j(u)-\int^u\mathrm{d}s\,A_i(s)A_i(s)\,. \tag{13}$$

[1] Alternatively, taking v to be time, rather than u, gives an autonomous system. However, to make connection to other cases it is more convenient if the wave depends on the choice of time. The same is often true in QED calculations [30,31].

[2] Defining $\mathcal{F}=\{Q_1,\ldots Q_N\}$ and following [27], the N quantities Q_j are functionally independent if the $N\times 8$ matrix $\mathcal{M}$ has rank N, where

$$\mathcal{M}_{l\mu}:=\left(\frac{\partial\mathcal{F}_l}{\partial x^\mu}, \frac{\partial\mathcal{F}_l}{\partial p_\mu}\right) \quad \text{(no sum)}. \tag{7}$$