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Contact geometry and quantum mechanics

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ABSTRACT

We present a generally covariant approach to quantum mechanics in which generalized positions, momenta and *time* variables are treated as coordinates on a fundamental "phase-spacetime". We show that this covariant starting point makes quantization into a purely geometric flatness condition. This makes quantum mechanics purely geometric, and possibly even topological. Our approach is especially useful for time-dependent problems and systems subject to ambiguities in choices of clock or observer. As a byproduct, we give a derivation and generalization of the Wigner functions of standard quantum mechanics.

1. Contact geometry

Mechanics is usually formulated in terms of an even $2n$-dimensional phase-space (or symplectic manifold) with time treated as an external parameter and dynamics determined by a choice of Hamiltonian. Yet classical physics ought not depend on choices of clocks. However, Einstein's principle of general covariance can be applied to this situation by introducing an odd $(2n+1)$-dimensional phase-spacetime manifold Z. Dynamics is now encoded by giving Z a (strict) contact structure—*i.e.*, a one-form α subject to a non-degeneracy condition on the (phase-spacetime) volume form:

$$\mathrm{Vol}_\alpha := \alpha \wedge (d\alpha)^{\wedge n} \neq 0\,. \tag{1.1}$$

Physical phase-spacetime trajectories γ are determined by extremizing the action

$$S = \int_\gamma \alpha\,. \tag{1.2}$$

Since the integral of a one-form along a path γ is a coordinate invariant quantity, general covariance (both worldline and target space) is built in from the beginning [1]. The equations of motion are

$$\varphi(\dot\gamma, \cdot) = 0\,, \tag{1.3}$$

where the two-form $\varphi := d\alpha$ is maximal rank by virtue of Eq. (1.1) and $\dot\gamma$ is a tangent vector to the path γ in Z.

The structure (Z, α) is called a (strict) contact geometry and Eq. (1.3) determines its Reeb dynamics [2]. In addition to general covariance, this formulation of mechanics enjoys a Darboux theorem, which implies the existence of *local* coordinates (ψ, π_A, χ^A) such that $\alpha = \pi_A d\chi^A - d\psi$ (where $A = 1, \ldots, n$) that trivialize the dynamics. Hence one might hope to treat classical *and* quantum mechanics as contact topology problems.

2. Goal

We aim to develop a generally phase-spacetime covariant formulation of quantum mechanics. We find a formulation of quantum mechanics in terms of intrinsic geometric structures on a contact manifold. Our approach is similar to Fedosov's quantization of symplectic manifolds [3], and indeed we were partly inspired by that work and subsequent applications of Fedosov quantization to models of higher spins [4]. Quantization based on contact geometry has been studied before: For example, Rajeev [5] considers quantization beginning with (classical) Lagrange brackets (the contact analog of Poisson brackets). Fitzpatrick [6] has extended this work to a rigorous geometric quantization setting. There is also earlier work by Kashiwara [7] that studies sheaves of pseudodifferential operators over contact manifolds. Investigations motivated by quantum cosmology of the so-called "clock ambiguity" in the quantum dynamics of time reparameterization invariant theories may be found in [8]. Contact geometry has also been employed in studies of choices of quantum clocks in [9].

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3. BRST analysis

Because it is worldline diffeomorphism invariant, the system with action (1.2) has one first class constraint. From the Darboux expression for the contact form α we see that there are also $2n$ second class constraints (the canonical momenta for the coordinates χ^A are constrained to equal the coordinates π_A). The quantization of constrained systems is well understood, thanks to the seminal work of Becchi, Rouet, Stora and Tyutin (BRST) [10]. We employ the Hamiltonian BRST technology of Batalin, Fradkin and Vilkovisky (BFV) [11] as well as its extension to systems with second class constraints [12]:

Let z^i be phase-spacetime coordinates and introduce canonical momenta p_i with Poisson brackets

$$\{z^i, p_j\}_{\rm PB} = \delta^i_j\,.$$

The second class constraints are

$$C_i = p_i - \alpha_i\,,$$

where $\alpha = \alpha_i dz^i$, $\varphi = \frac{1}{2}\varphi_{ij} dz^i \wedge dz^j$, and

$$\{C_i, C_j\}_{\rm PB} = \varphi_{ij}\,.$$

Second class constraints require Dirac brackets; alternatively one may introduce $2n$ new variables s^a with Poisson brackets

$$\{s^a, s^b\}_{\rm PB} = J^{ab}\,,$$

where J is a constant, maximal rank, $2n \times 2n$ matrix [12]. At least locally, we can introduce $2n$ linearly independent soldering forms e^a (analogous to the vielbeine/tetrads of general relativity) such that

$$\varphi = \frac{1}{2} J_{ab} e^a \wedge e^b\,,$$

and $J_{ab} J^{bc} = \delta^c_a$. In these terms our system is now described by an extended action functional subject only to $2n+1$ first class constraints:

$$S_{\rm ext}[z(\tau), s(\tau)] = \int \Big[\tfrac{1}{2} s^a J_{ab} \dot s^b + \dot z^i\big(\alpha_i(z) + s^a J_{ab} e_i{}^b(z) + \omega_i(z,s)\big)\Big] d\tau\,. \quad (3.1)$$

In the above, τ is an arbitrary choice of worldline parameter, and the s-dependent one-form $\omega(z,s)$ on Z must be chosen to obey[1]

$$d\Omega + \tfrac{1}{2}\{\Omega \wedge \Omega\}_{\rm PB} = 0\,,$$

where $\Omega = \alpha + s^a J_{ab} e^b + \omega$, in order that the extended constraints $C^{\rm ext}_i = p_i - \Omega_i$ are first class. Locally, the Darboux theorem implies that a set of one-forms e^a with a flat connection exists.

The gauge invariances

$$\delta z^i = \varepsilon^i(\tau)\,, \qquad \delta s^a = \varepsilon^i(\tau) J^{ab} \frac{\partial \Omega_i}{\partial s^b}\,,$$

ensure[2] that the equations of motion

$$J_{ab} \dot s^b + \dot z^i \frac{\partial \Omega_i}{\partial s^a} = 0 = \dot z^i\big(\partial_i \Omega_j - \partial_j \Omega_i\big) - \dot s^a \frac{\partial \Omega_j}{\partial s^a}\,,$$

are equivalent to Reeb dynamics.

[1] Note that $\{\Omega \wedge \Omega\}_{\rm PB} := dz^i \wedge dz^j \{\Omega_i, \Omega_j\}_{\rm PB}$. In related work, the authors of [14] have constructed a flat Cartan Maurer connection from a central extension of the group of canonical transformations.

[2] Here we assume that the rectangular matrix $\frac{\partial \Omega_i}{\partial s^b}$ has maximal rank, which is guaranteed at least in a neighborhood of $s = 0$.

Now that we are dealing with a first class constrained system, the BFV quantum action follows directly

$$S_{\rm qu} = \int \big[\Theta + \{Q, \Phi\}_{\rm PB}\big]\,.$$

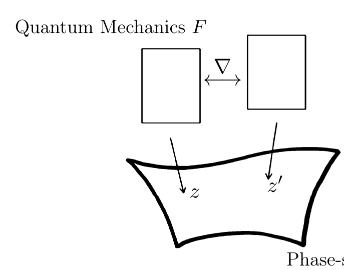
Here Φ is the gauge fixing fermion for some choice of gauge and Θ is the BRST-extended symplectic current

$$\Theta = p_i \dot z^i + \frac{1}{2} s^a J_{ab} \dot s^b + b_i \dot c^i\,,$$

where (b_i, c^i) are canonically conjugate Grassmann ghosts. The BRST charge $Q = c^i C^{\rm ext}_i$ is determined by the first class constraints.

4. Quantization

We are now ready to quantize the contact formulation of classical mechanics. The physical picture underlying our method closely mimics general relativity: Spinors in curved space are described by gluing a copy of a flat space Clifford algebra and its spin representation to each point in spacetime using vielbeine and the spin connection to compare spinors at differing spacetime points. Mathematically, this is an example of a vector bundle in which context vielbeine are called soldering forms. Here we want to glue a copy of standard quantum mechanics to each point z in the phase-spacetime Z, which we view as the fibers of a suitable vector bundle, and then construct a connection ∇ to compare differing fibers, as depicted below:



In this picture, quantum mechanics along the fibers is described in terms of the variables s^a which are quantized in the standard way by choosing some polarization in which

$$\hat s^a = \Big(S^A, \frac{\hbar}{i}\frac{\partial}{\partial S^A}\Big)\,.$$

Quantum wavefunctions $\Psi(S^A)$ depend on half the s-variables S^A spanning $\mathbb{R}^n$, and the inner product is the usual one: $\langle \Psi, \Psi' \rangle = \int_{\mathbb{R}^n} \Psi^* \Psi'$. The "Schrödinger equation" along each fiber as well as the parallel transport of quantum mechanics from fiber to fiber is controlled by the connection ∇ given by the quantum BRST charge $\widehat{Q}$. To compute this connection, we quantize the contact coordinates z^i and their momenta using the polarization

$$\hat p_i = \frac{\hbar}{i}\frac{\partial}{\partial z^i}\,.$$

In addition, we identify the Grassmann ghosts c^i with a basis of one-forms dz^i along Z. Hence BRST wavefunctions now depend on (z^i, dz^i, S^A) and may be viewed as differential forms on the contact manifold Z taking values in the Hilbert space $L^2(\mathbb{R}^n)$. The quantum BRST charge $\widehat{Q} = \frac{\hbar}{i}\nabla$ where ∇ is the operator-valued connection[3]

[3] In fact, exactly such a connection over a symplectic manifold has been introduced in [13].