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Frame-dragging effect in the field of non rotating body due to unit gravimagnetic moment

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ABSTRACT

Nonminimal spin-gravity interaction through unit gravimagnetic moment leads to modified Mathisson–Papapetrou–Tulczyjew–Dixon equations with improved behavior in the ultrarelativistic limit. We present exact Hamiltonian of the resulting theory and compute an effective $\frac{1}{c^2}$-Hamiltonian and leading post-Newtonian corrections to the trajectory and spin. Gravimagnetic moment causes the same precession of spin $\mathbf{S}$ as a fictitious rotation of the central body with angular momentum $\mathbf{J} = \frac{M}{m}\mathbf{S}$. So the modified equations imply a number of qualitatively new effects, that could be used to test experimentally, whether a rotating body in general relativity has null or unit gravimagnetic moment.

The manifestly generally covariant Mathisson–Papapetrou–Tulczyjew–Dixon (MPTD) equations [1–6] are widely used to describe a rotating test body in general relativity in pole–dipole approximation. In the current literature (see [7–12] and references therein), they usually appear in the form given by Dixon

$$\nabla P_\mu = -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu, \qquad \nabla S^{\mu\nu} = 0, \tag{1}$$

where $\theta_{\mu\nu} = R_{\mu\nu\alpha\beta}S^{\alpha\beta}$ is the gravitational analogy of the electromagnetic field strength $F_{\mu\nu}$ [7,13]. (Our spin-tensor $S^{\mu\nu}$ is twice of that of Dixon. Besides, in the last equation we omitted the term $2P^{[\mu}\dot{x}^{\nu]}$, which does not contribute in $\frac{1}{c^2}$-approximation we are interested in the present work. Concerning other notation, see the footnote.[1]) Together with the spin supplementary condition[2]

$$S^{\mu\nu}P_\nu = 0, \tag{2}$$

MPTD equations prescribe the evolution of both trajectory and spin of the body in $1/c^2$-approximation.

Starting from the pioneer works, MPTD equations were considered as a Hamiltonian-type system. Following this spirit in the recent work [15], we explicitly realized this idea by constructing the minimal interaction with gravity in the Lagrangian of vector model of spinning particle, and showed that this indeed leads to MPTD equations in the Hamiltonian formalism. This allowed us to study the ultrarelativistic limit in exact equations for the trajectory of MPTD particle in the laboratory time. Using the Landau–Lifshitz $(1+3)$-decomposition [16] we observed that, unlike a geodesic equation, the MPTD equations lead to the expression for three-acceleration which contains divergent terms as $v \to c$ [13]. Therefore it seems interesting to find a generalization of MPTD equations with improved behavior in the ultrarelativistic regime. This can be achieved, if we add a nonminimal spin-gravity interaction through gravimagnetic moment κ [13]. $\kappa = 0$ corresponds to the MPTD equations. The most interesting case turns out to be $\kappa = 1$ (gravimagnetic body). Keeping only the terms, which may contribute in the leading post-Newtonian approximation, this gives the modified equations (among other equations, see below)

$$\nabla P_\mu = -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu - \frac{\sqrt{-\dot{x}^2}}{32mc}(\nabla_\mu\theta_{\sigma\lambda})S^{\sigma\lambda},$$

$$\nabla S^{\mu\nu} = \frac{\sqrt{-\dot{x}^2}}{4mc}\theta^{[\mu}{}_\alpha S^{\nu]\alpha}. \tag{3}$$

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[1] Our variables are taken in arbitrary parametrization τ, then $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$ and the covariant derivative is $\nabla\omega^\mu = \frac{d\omega^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\omega^\beta$. The square brackets mean antisymmetrization, $\omega^{[\mu}\pi^{\nu]} = \omega^\mu\pi^\nu - \omega^\nu\pi^\mu$. We often miss the four-dimensional indexes and use the notation $\dot{x}^\mu N_{\mu\nu}\dot{x}^\nu = \dot{x}N\dot{x}$, $N^\mu{}_\nu\dot{x}^\nu = (N\dot{x})^\mu$, $\omega^2 = g_{\mu\nu}\omega^\mu\omega^\nu$, $\mu, \nu = 0, 1, 2, 3$, $\operatorname{sign} g_{\mu\nu} = (-, +, +, +)$. Suppressing the indexes of three-dimensional quantities, we use bold letters. The tensor of Riemann curvature is $R^\sigma{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\sigma{}_{\lambda\nu} - \partial_\nu\Gamma^\sigma{}_{\lambda\mu} + \Gamma^\sigma{}_{\beta\mu}\Gamma^\beta{}_{\lambda\nu} - \Gamma^\sigma{}_{\beta\nu}\Gamma^\beta{}_{\lambda\mu}$.

[2] While the Lagrangian and Hamiltonian formalisms dictate [14] the condition (2), in the multipole approach there is a freedom in the choice of a spin supplementary condition, related with the freedom to choose a representative point of the body [3,4,6]. Different conditions lead to the same results for observables in $\frac{1}{c^2}$-approximation, see [36,5,42].

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Comparing (3) with (1), we see that unit gravimagnetic moment yields quadratic in spin corrections to MPTD equations in $\frac{1}{c^2}$-approximation.

Both acceleration and spin torque of gravimagnetic body have reasonable behavior in ultrarelativistic limit [13]. In the present work we study the modified equations and the corresponding effective Hamiltonian in the regime of small velocities, and compute $\frac{1}{c^2}$-corrections due to the extra-terms appeared in (3). In Schwarzschild and Kerr space–times, the modified equations predict a number of qualitatively new effects, that could be used to test experimentally, whether a rotating body in general relativity has null or unit gravimagnetic moment.

Let us briefly describe the variational problem which implies the modified equations (3). In the vector model of spin presented in [17], the configuration space consist of the position of the particle $x^\mu(\tau)$, and the vector $\omega^\mu(\tau)$ attached to the point $x^\mu(\tau)$. Minimal interaction with gravity is achieved by direct covariantization of the free action, initially formulated in Minkowski space. That is we replace $\eta_{\mu\nu} \to g_{\mu\nu}$, and usual derivative of the vector ω^μ by the covariant derivative: $\dot\omega^\mu \to \nabla\omega^\mu$. The nonminimal spin-gravity interaction through the gravimagnetic moment κ can be thought as a deformation of original metric: $g^{\mu\nu} \to \sigma^{\mu\nu} = g^{\mu\nu} + \kappa R_\alpha{}^\mu{}_\beta{}^\nu \omega^\alpha\omega^\beta$, with the resulting Lagrangian action [13]

$$S = -\int d\tau \sqrt{(mc)^2 - \frac{\alpha}{\omega^2}} \times \sqrt{-\dot{x}NK\sigma N\dot{x} - \nabla\omega NKN\nabla\omega + 2\lambda\dot{x}NKN\nabla\omega}. \quad (4)$$

We have denoted $K = (\sigma - \lambda^2 g)^{-1}$, where λ is the only Lagrangian multiplier in the theory. The matrix $N_{\mu\nu} \equiv g_{\mu\nu} - \frac{\omega_\mu\omega_\nu}{\omega^2}$ is a projector on the plane orthogonal to ω: $N_{\mu\nu}\omega^\nu = 0$. The parameter α determines the value of spin, in particular, $\alpha = \frac{3\hbar^2}{4}$ corresponds to the spin one-half particle. In the spinless limit, $\omega^\mu = 0$ and $\alpha = 0$, Eq. (4) reduces to the standard Lagrangian of a point particle, $-mc\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$.

The action (4) is manifestly invariant under general-coordinate transformations as well as under reparametrizations of the evolution parameter τ. Besides, there is one more local symmetry, which acts in the spin-sector and called the spin-plane symmetry: the action remains invariant under rotations of the vectors ω^μ and $\pi_\mu = \frac{\partial L}{\partial\dot\omega^\mu}$ in their own plane [18]. Being affected by the local transformation, these vectors do not represent observable quantities. But their combination

$$S^{\mu\nu} = 2(\omega^\mu\pi^\nu - \omega^\nu\pi^\mu) = (S^{i0} = D^i,\ S_{ij} = 2\epsilon_{ijk}S_k), \quad (5)$$

is an invariant quantity, which represents the spin-tensor of the particle. In Eq. (5), we decomposed the spin-tensor into three-dimensional spin-vector $\mathbf{S} = \frac{1}{2}(S^{23}, S^{31}, S^{12})$, and dipole electric moment [19] D^i.

For the general-covariant and spin-plane invariant variables x^μ, $P_\mu = p_\mu - \Gamma^\beta_{\alpha\mu}\omega^\alpha\pi_\beta$ and $S^{\mu\nu}$ (here $p_\mu = \frac{\delta S}{\delta\dot{x}^\mu}$), the Hamiltonian equations of motion of the theory (4) acquire especially simple form when $\kappa = 1$. In $\frac{1}{c^2}$-approximation, we obtained the equations (3), accompanied by the Hamiltonian equation for x^μ, $\dot{x}^\mu = \frac{\sqrt{-\dot{x}^2}}{mc}P_\mu$, the latter can be identified with velocity–momentum relation implied by MPTD equations [13]. Besides the dynamical equations, the variational problem (4) implies the mass-shell constraint

$$T \equiv P^2 + \frac{\kappa}{16}\theta_{\mu\nu}S^{\mu\nu} + (mc)^2 = 0, \quad (6)$$

and the spin-sector constraints $P\omega = 0$, $P\pi = 0$, $\omega\pi = 0$ and $\pi^2 - \frac{\alpha}{\omega^2} = 0$. Their meaning becomes clear if we consider their effect over the spin-tensor. The second-class constraints $P\omega = 0$ and $P\pi = 0$ imply the spin supplementary condition (2), while the remaining first-class constraints fix the value of square of the spin-tensor, $S^{\mu\nu}S_{\mu\nu} = 8\alpha$. The equations imply that only two components of spin-tensor are independent, as it should be for an elementary spin one-half particle. The mass-shell constraint (6) look like that of a spinning particle with gyromagnetic ratio g, $P^2 - \frac{eg}{c}F_{\mu\nu}S^{\mu\nu} + (mc)^2 = 0$. In view of this similarity, the interaction constant κ has been called gravimagnetic moment [20,7].

Although the vector model of spin has been initially developed to describe an elementary particle of spin one-half, it can be adopted to study a rotating body in general relativity. The action (4) with $\kappa = 0$ implies MPTD equations, and the only difference among two formalisms is that values of momentum and spin are conserved quantities of MPTD equations, while in the vector model they are fixed by constraints. In summary [13], to study the class of trajectories of a body with $\sqrt{-P^2} = k$ and $S^2 = \beta$, we can use our spinning particle with $m = \frac{k}{c}$ and $\alpha = \frac{\beta}{8}$.

Although the post-Newtonian approximation can be obtained by direct computations on the base of equations of motion, we prefer to work with an approximate Hamiltonian. This gives a more transparent picture of nonminimal interaction, in particular, display strong analogy with a spinning particle with magnetic moment in electromagnetic background. We could consider a Hamiltonian corresponding to either Poisson or Dirac brackets. We work with Dirac bracket[3] for the second-class constraints $P\omega = 0$ and $P\pi = 0$, since in this case the relativistic Hamiltonian acquires the conventional form $H_{rel} = \frac{\lambda}{2}T$. According to the procedure described in [21], exact Hamiltonian for dynamical variables $\mathbf{x}(t)$, $\mathbf{p}(t)$ and $\mathbf{S}(t)$ as functions of the coordinate time $t = \frac{x^0}{c}$ is $H = -cp_0$, where the conjugated momentum p_0 is a solution to the mass-shell constraint (6). Solving the constraint, we obtain

$$H = \frac{c}{\sqrt{-g^{00}}}\sqrt{(mc)^2 + \gamma^{ij}P_iP_j + \frac{1}{16}(\theta S)} - c\pi_\mu\Gamma^\mu{}_{0\nu}\omega^\nu + \frac{cg^{0i}}{g^{00}}P_i, \quad (7)$$

where $\gamma^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}$. Let us consider a stationary, asymptotically flat metric of a spherical body with mass M and angular momentum $\mathbf{J}$. In the post-Newtonian approximation up to order $\frac{1}{c^4}$, this reads [26]

$$ds^2 = \left(-1 + \frac{2GM}{c^2r} - \frac{2G^2M^2}{c^4r^2}\right)(dx^0)^2 - 4G\frac{\epsilon_{ijk}J^jx^k}{c^3r^3}dx^0dx^i + \left(1 + \frac{2GM}{c^2r} + \frac{3G^2M^2}{2c^4r^2}\right)dx^idx^i. \quad (8)$$

To obtain the effective Hamiltonian, we expand all quantities in (7) in series up to $\frac{1}{c^2}$-order. To write the result in a compact form, we introduce the vector potential $\mathbf{A}_J = \frac{2G}{c}[\mathbf{J} \times \frac{\mathbf{r}}{r^3}]$ for the gravitomagnetic field $\mathbf{B}_J = [\nabla \times \mathbf{A}_J] = \frac{2G}{c}\frac{3(\mathbf{J}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{J}}{r^3}$, produced by rotation of central body (we use the conventional factor $\frac{2G}{c}$, different from that of Wald [31]. In the result, our $\mathbf{B}_J = 4\mathbf{B}_{Wald}$). Besides we define the vector potential $\mathbf{A}_S = \frac{M}{m}\frac{G}{c}[\mathbf{S} \times \frac{\mathbf{r}}{r^3}]$ of fictitious gravitomagnetic field $\mathbf{B}_S = [\nabla \times \mathbf{A}_S] = \frac{M}{m}\frac{G}{c}\frac{3(\mathbf{S}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{S}}{r^3}$ due to rotation of a gyroscope,

[3] The Dirac bracket turns the spinning particle into intrinsically noncommutative theory. This could manifest itself in various applications [22–24]. In particular, our Hamiltonian differs from those suggested by other groups, for instance [25]. They have been compared in [15].