Physics Letters B 776 (2018) 1–4

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

# On the number of light rings in curved spacetimes of ultra-compact objects

Shahar Hod [a,b,*]

[a] The Ruppin Academic Center, Emeq Hefer 40250, Israel
[b] The Hadassah Academic College, Jerusalem 91010, Israel

ARTICLE INFO

*Article history:*
Received 29 September 2017
Received in revised form 8 November 2017
Accepted 10 November 2017
Available online 14 November 2017
Editor: M. Cvetič

ABSTRACT

In a very interesting paper, Cunha, Berti, and Herdeiro have recently claimed that ultra-compact objects, self-gravitating horizonless solutions of the Einstein field equations which have a light ring, must possess at least *two* (and, in general, an even number of) light rings, of which the inner one is *stable*. In the present compact paper we explicitly prove that, while this intriguing theorem is generally true, there is an important exception in the presence of degenerate light rings which, in the spherically symmetric static case, are characterized by the simple dimensionless relation $8\pi r_\gamma^2(\rho + p_{\rm T}) = 1$ [here $r_\gamma$ is the radius of the light ring and $\{\rho, p_{\rm T}\}$ are respectively the energy density and tangential pressure of the matter fields]. Ultra-compact objects which belong to this unique family can have an *odd* number of light rings. As a concrete example, we show that spherically symmetric constant density stars with dimensionless compactness $M/R = 1/3$ possess only *one* light ring which, interestingly, is shown to be *unstable*.



## 1. Introduction

One of the most physically interesting predictions of general relativity is the existence of closed light rings in curved spacetimes of compact astrophysical objects. These null circular geodesics are usually associated with black-hole spacetimes (see [1–5] and references therein), but horizonless compact objects like boson stars may also possess light rings [6–8].

Intriguingly, Cunha, Berti, and Herdeiro [6] (see also [7]) have recently asserted that horizonless matter configurations that have a light ring (the term ultra-compact objects is usually used in the literature to describe these self-gravitating field configurations) must have *pairs* (that is, an even number) of light rings. In particular, the interesting claim has been made [6,7] that, for these regular ultra-compact horizonless objects, one of the closed light rings is *stable* [9].

Combining this claimed property of ultra-compact objects with the intriguing suggestion made in [11] (see also [3,4]) that, due to the fact that massless perturbation fields tend to pile up on stable null geodesics, curved spacetimes with stable light rings may develop non-linear instabilities, it has been argued [4,6,7] that horizonless ultra-compact objects are non-linearly unstable and thus cannot provide physically acceptable alternatives to the canonical black-hole solutions of the Einstein field equations [12].

The main goal of the present paper is to prove that, while the physically intriguing theorem presented in [6] is generally true, it may be violated by ultra-compact objects with degenerate null circular geodesics which, in the spherically symmetric case, are characterized by the simple dimensionless relation $8\pi r_\gamma^2(\rho + p_{\rm T}) = 1$ [13]. In particular, below we shall explicitly demonstrate that, in principle, one can have special horizonless ultra-compact objects which possess only *one* light ring which, interestingly, is shown to be *unstable*.

## 2. Description of the system

We consider spherically symmetric nonlinear matter configurations which are characterized by the static line element [1–5,14,15]

$$ds^2 = -e^{-2\delta}\mu dt^2 + \mu^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \; , \tag{1}$$

where $\delta = \delta(r)$ and $\mu = \mu(r)$. Regular spacetimes are characterized by the near-origin behavior [4]

$$\mu(r \to 0) = 1 + O(r^2) \quad \text{and} \quad \delta(0) < \infty \; . \tag{2}$$

* Correspondence to: The Ruppin Academic Center, Emeq Hefer 40250, Israel.
*E-mail address:* shaharhod@gmail.com.

https://doi.org/10.1016/j.physletb.2017.11.021

In addition, asymptotically flat matter configurations are characterized by the functional relations [4]

$$\mu(r\to\infty)\to 1 \quad \text{and} \quad \delta(r\to\infty)\to 0\ . \tag{3}$$

For spherically symmetric static spacetimes, the composed Einstein-matter field equations, $G^{\mu}_{\nu}=8\pi T^{\mu}_{\nu}$, are given by [4,16]

$$\mu'=-8\pi r\rho+\frac{1-\mu}{r} \tag{4}$$

and

$$\delta'=-\frac{4\pi r(\rho+p)}{\mu}\ , \tag{5}$$

where $(\rho, p, p_{\rm T})\equiv(-T^{t}_{t}, T^{r}_{r}, T^{\theta}_{\theta}=T^{\phi}_{\phi})$ are respectively the energy density, the radial pressure, and the tangential pressure of the horizonless matter configuration [17]. We shall assume that the matter fields satisfy the dominant energy condition [18]

$$\rho\geq|p|,|p_T|\geq 0\ . \tag{6}$$

The gravitational mass $m(r)$ of the field configuration contained within a sphere of areal radius $r$ is given by the integral relation [4,19]

$$m(r)=4\pi\int_0^r x^2\rho(x)dx\ . \tag{7}$$

The relations (6) and (7) imply that regular matter configurations with finite ADM masses (as measured by asymptotic observers) are characterized by the asymptotic functional behavior

$$r^3p(r)\to 0 \quad \text{as} \quad r\to\infty\ . \tag{8}$$

Defining the dimensionless radial function

$$\mathcal{R}(r)\equiv 3\mu-1-8\pi r^2 p\ , \tag{9}$$

one finds that the conservation equation

$$T^{\mu}_{r;\mu}=0 \tag{10}$$

together with the Einstein differential equations (4) and (5) yield, for spherically symmetric static spacetimes, the radial pressure gradient

$$p'(r)=\frac{1}{2\mu r}\big[\mathcal{R}(\rho+p)+2\mu T-8\mu p\big]\ , \tag{11}$$

where

$$T=-\rho+p+2p_T \tag{12}$$

is the trace of the energy–momentum tensor which characterizes the self-gravitating matter fields.

## 3. The number of light rings of spherically symmetric ultra-compact objects

In the present section we shall explicitly prove that, while the intriguing Cunha–Berti–Herdeiro theorem [6] is generally true, it may be violated by a special family of horizonless ultra-compact objects that have a light ring which is characterized by the dimensionless functional relation $8\pi r^2_{\gamma}(\rho+p_{\rm T})=1$.

To this end, we shall first derive, following the analysis of [1,3,4], the characteristic functional relation of null circular geodesics (light rings) in the spherically symmetric static spacetime (1). Taking cognizance of the fact that the curved line element (1) is independent of the time and angular coordinates $\{t,\phi\}$, one deduces that the geodesic trajectories are characterized by two conserved physical quantities: the energy $E$ and the angular momentum $L$ [1,3,4]. In particular, the null geodesics of the spherically symmetric static spacetime (1) are governed by an effective radial potential $V_r$ which satisfies the characteristic equation [1,3,4,20]

$$E^2-V_r\equiv\dot{r}^2=\mu\Big(\frac{E^2}{e^{-2\delta}\mu}-\frac{L^2}{r^2}\Big)\ . \tag{13}$$

The characteristic circular geodesics of the horizonless curved spacetime (1) are determined by the effective radial potential (13) through the relations [1,3,4,21]

$$V_r=E^2 \quad \text{and} \quad V'_r=0\ . \tag{14}$$

Taking cognizance of Eqs. (4), (5), and (13), one can express (14) in the simple dimensionless form [1,3,4]

$$\mathcal{R}(r=r_{\gamma})=0\ . \tag{15}$$

The remarkably compact functional relation (15) determines the characteristic null circular geodesics of the spherically symmetric static spacetime (1). For later purposes we note that one learns from Eqs. (2), (3), and (8) that the dimensionless radial function $\mathcal{R}(r)$ [see Eq. (9)] is characterized by the simple relations

$$\mathcal{R}(r=0)=2 \quad \text{and} \quad \mathcal{R}(r\to\infty)\to 2\ . \tag{16}$$

As explicitly shown in [1,3], stable circular geodesics of the spherically symmetric static spacetime (1) are characterized by a locally convex effective radial potential with the property $V''_r(r=r_{\gamma})>0$, whereas unstable circular geodesics are characterized by a locally concave effective radial potential with the opposite property $V''_r(r=r_{\gamma})<0$. Using Eqs. (4), (5), (9) and (11), one finds the compact functional relation

$$V''_r(r=r_{\gamma})=-\frac{E^2e^{2\delta}}{\mu r_{\gamma}}\times\mathcal{R}'(r=r_{\gamma}) \tag{17}$$

for the second spatial derivative of the effective radial potential (13), where [see Eqs. (4), (9), and (11)]

$$\mathcal{R}'(r=r_{\gamma})=\frac{2}{r_{\gamma}}\big[1-8\pi r^2_{\gamma}(\rho+p_T)\big]\ . \tag{18}$$

Taking cognizance of Eqs. (15) and (16), one deduces that horizonless ultra-compact objects are generally (but, as will be discussed below, not always) characterized by a discrete set $\{r_{\gamma1},r_{\gamma2},...,r_{\gamma2n-1},r_{\gamma2n}\}$ of *even* number of light rings with the property $\mathcal{R}(r=r_{\gamma i})=0$. In particular, the subset $\{r_{\gamma1},r_{\gamma3},...,r_{\gamma2n-1}\}$ of odd light rings is generally characterized by the property $\mathcal{R}'(r=r_{\gamma i})<0$, whereas the subset $\{r_{\gamma2},r_{\gamma4},...,r_{\gamma2n}\}$ of even light rings is generally characterized by the property $\mathcal{R}'(r=r_{\gamma i})>0$. Thus, the first subset corresponds to stable light rings with the property $V''_r(r=r_{\gamma})>0$ [see Eq. (17)], whereas the second subset corresponds to unstable light rings with the property $V''_r(r=r_{\gamma})<0$. It is important to stress the fact that this conclusion agrees with the recently published interesting theorem of Cunha, Berti, and Herdeiro [6].

However, it is also physically important to stress the fact that there are special cases in which one (or more) of the light rings, $r=r^*_{\gamma}$, which characterize the horizonless ultra-compact objects is characterized by the functional relation [22]

$$\mathcal{R}'(r=r^*_{\gamma})=0\ . \tag{19}$$

In this case the horizonless ultra-compact objects may be characterized by an *odd* number of light rings.

In particular, as a special case that will be demonstrated explicitly in the next section, we note that there are ultra-compact